MST204.8

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 8 Damped and forced vibrations

The Open University

Mathematics/Science/Technology
An Inter-faculty Second Level Course

MST204 Mathematical Models and Methods

Unit 8
Damped and forced vibrations

Prepared by the Course Team

The Open University

The Open University, Walton Hall, Milton Keynes MK7 6AA.

First published 1982. New edition 1990. Reprinted 1992, 1994, 1996.

Designed by the Graphic Design Group of the Open University.

Typeset by the $\mathrm{T_EX}$project, Faculty of Mathematics, The Open University.

Printed and bound in the United Kingdom by Staples Printers Rochester Limited,
Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT.

ISBN 0 7492 2032 5

2.4

Contents

Introduction 4

Study guide 4

1 Damped vibrations 5

1.1 Introduction 5

1.2 Linear damping 6

1.3 The revised equation of motion 6

1.4 The condition for oscillation 10

1.5 Weak, critical and strong damping 13

Summary of Section 1 15

2 Forced vibrations 16

2.1 Introduction 16

2.2 The equation of motion 16

2.3 The damping ratio 19

2.4 Resonance 23

Summary of Section 2 24

3 The perfect dashpot 25

3.1 Introducing the dashpot 25

3.2 Deriving equations of motion (Audio-tape Subsection) 29

Summary of Section 3 33

4 Off the record: resonance and damping (Television Section) 33

5 End of unit exercises 38

Appendix: Solutions to the exercises 39

Introduction

In *Unit 7* you studied a model of vibrating systems which is based upon the concept of a perfect spring and leads to the prediction of simple harmonic motion. This unit continues the theme of modelling vibration, introducing the three ideas of *damping*, *forcing* and *resonance*.

Damping refers to the inevitable tendency of vibrations to die away whenever a system is left to its own devices. This phenomenon was clearly apparent in the oscillations of the home-made apparatus of *Unit 7* Section 1, but no attempt was made there to include damping in the model. In this unit we model damped vibrations by including a velocity-dependent resistive force in Newton's second law.

Forcing is what is needed to keep a system vibrating, in spite of its damping. This can arise from a periodic force. For example, a car engine vibrates due to the rotating crank even when the car is stationary. A mechanical system can also be forced to vibrate by the specific movement of some component of the system. Thus a car on a bumpy road is forced to vibrate, at a frequency not of its own choosing, by the bumps which its wheels encounter. Similarly, the tone-arm of a record player responds to the motion of the stylus in contact with a record.

Resonance refers to the fact that the amplitude of forced vibrations can be much greater at some forcing frequencies than at others. These resonant oscillations can be destructive in mechanical systems. Often designers introduce springs and damping devices in order to eliminate resonant behaviour over the range of frequencies which a piece of machinery is likely to encounter.

The mathematics required here is that which you studied in *Unit 6*. In particular, the mechanical systems to be considered are modelled by the linear differential equation

$$m\ddot{x} + r\dot{x} + kx = kx_e + P\cos\omega t,$$

where m, r, k, x_e, P and ω are constants. The case for which $r = 0$ and $P = 0$ was discussed in *Unit 7*, and corresponds to simple harmonic motion. Damped vibrations are modelled by the addition of the term $r\dot{x}$, where r is a positive constant. The inclusion of the term $P\cos\omega t$ corresponds to forced vibrations.

Study guide

The first four sections of this unit should be studied in the order that they appear. The main ideas are in Sections 1 and 2. Section 1 uses Newton's second law, with perfect spring forces and resistive forces which are linearly dependent on velocity, to model *damped* vibrations. A periodic applied force is added to this model in Section 2 in order to incorporate *forced* vibrations. This basic model is generalized in two respects in Section 3. First, damping forces are discussed which depend on the *relative* velocity between two components of a system. Secondly, we consider forced vibrations which arise from the periodic movement of a point of the system other than the particle whose motion is sought.

The audio-tape is associated with Section 3. It will give you practice in using Newton's second law to derive the equations of motion which model mechanical systems. The television programme is associated with Section 4 and provides a basis for evaluating the success of our models of damped and forced vibrations. It will be easier to follow if you have already studied the first three sections.

Finally, Section 5 contains some further exercises on the whole unit. You may use these for additional practice, either during the current study week or when you are revising for the examination.

1 Damped vibrations

1.1 Introduction

In *Unit 7* we modelled a variety of vibrating systems, and in each case the model led to the prediction of *simple harmonic motion*. But in the real world no system, left to its own devices, maintains the oscillations of constant amplitude that are predicted by such models; in reality the amplitude *decreases* with time. In this section we therefore introduce a new ingredient to model the observed behaviour of *damped vibration*. This ingredient is a *velocity-dependent* force.

One particular system considered in *Unit 7* was a home-made apparatus used in a simple experiment. The system consisted of a bag of coins suspended from an elastic string whose other end was attached to a fixed frame. The bag was pulled down below its position of rest and then released, after which it oscillated up and down between extreme positions on either side of its equilibrium position. The mathematical model of *Unit 7*, used in an attempt to describe this behaviour, predicts that when the displacement of the bag of coins is plotted against time, the resulting curve will be sinusoidal, as shown in Figure 1.

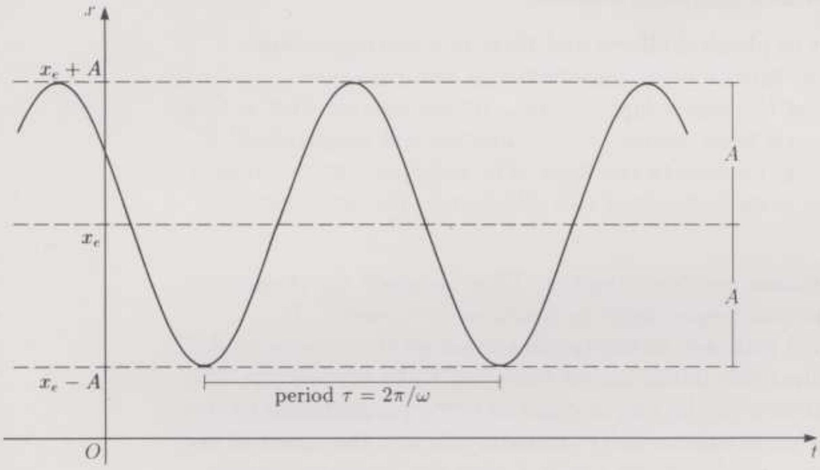


Figure 1 Simple harmonic motion with amplitude A and period τ.

Specifically, the model predicts that the equation of motion is $\ddot{x} + \omega^2 x = \omega^2 x_e$ and that the displacement of the bag is given by $x = x_e + A\cos(\omega t + \phi)$, where x_e is the equilibrium displacement, A is the amplitude, ω is the angular frequency and ϕ is the phase angle.

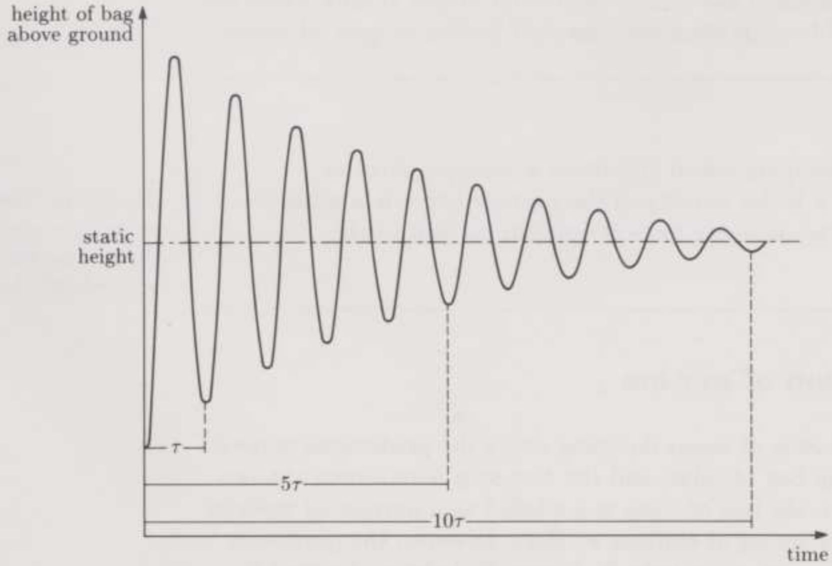


Figure 2 The observed behaviour of the home-made apparatus of *Unit 7* Section 1, showing damped vibration.

For this simple harmonic motion both the amplitude and the period $\tau = 2\pi/\omega$ are constant. How does this prediction tally with observed fact? The experimental results shown in Figure 3 of *Unit 7* Section 1 supply the answer. The essential features of this observed motion are repeated in Figure 2 above.

In one respect the prediction corresponds with what is observed: the motion does have a constant period. But from another point of view the prediction is clearly wrong: the motion is *not* one with a constant amplitude. In fact, the amplitude decreases steadily with time and eventually the motion ceases altogether. This observed decrease in amplitude is not represented in the simple harmonic model, and in the current section we intend to remedy this deficiency.

1.2 Linear damping

It is a fair guess that a major reason for the decreasing amplitude of the vibration represented in Figure 2 is the effect of air resistance. However, it is known that the repeated lengthening and shortening of a rubber band causes an energy loss due to internal friction, and this will also contribute to the decrease in amplitude. In the context of vibration, frictional effects are commonly referred to as *damping* and the corresponding motion is spoken of as a damped vibration.

Damping can be due to a number of physical effects and there is a corresponding variety of mathematical models. In this unit we concentrate on the resistance experienced by a solid body moving through a liquid or gas. As you saw in *Unit 4*, this resistance is commonly modelled as a force opposed to the motion and proportional to some power of the speed of the body relative to the fluid. The resistance to the motion of the bag of coins through the air is an example of this effect, as is the air resistance acting on moving vehicles.

In *Unit 4* two models for air resistance were investigated. They assumed the magnitude of the resistive force to be proportional respectively to speed, or to $(\text{speed})^2$. In mathematical modelling it is a good principle to start with the simplest available model, and not to complicate matters unless this initial model turns out to be inadequate. We shall therefore model the air resistance on the bag of coins as being proportional to the speed of the bag (which, since the air is substantially stationary, is also the speed of the bag relative to the air). Thus if the velocity of the bag is $\dot{x}$ then the damping force is assumed to have magnitude $R = r|\dot{x}|$, where r is a positive constant, and direction opposing the motion. This is a much used model called **linear damping**. It has the virtue of simplicity and is mathematically easy to handle, which is such an advantage that the model is sometimes used even in situations where it may not be strictly appropriate. Any part of the frictional force on the bag of coins which is due to effects other than air resistance (such as internal friction in the elastic string) is a case in point. In such circumstances one would not expect the model to give results which are accurate in detail, but it is capable of showing the effects of friction in general terms.

Linear damping

We use a linear model for damping which stipulates a damping force of magnitude $R = r|\dot{x}|$, where $\dot{x}$ is the velocity of the particle and r is a positive constant. The direction of the damping force is opposite to that of the velocity.

The SI units of the damping constant r are $\mathrm{N\,m^{-1}\,s}$ (force divided by speed), or equivalently $\mathrm{kg\,s^{-1}}$.

1.3 The revised equation of motion

We now investigate how the inclusion of linear damping alters the predictions derived from our model for the oscillating bag of coins, and the first step is to obtain the new equation of motion. As in *Unit 7*, the bag of coins is modelled as a particle of mass m and the elastic string as a perfect spring of stiffness k. Here, however, the particle is also acted upon by a damping force of magnitude R. Figure 3(a) shows the model without specifying any forces. Its purpose is to establish the origin for the displacement

x of the particle. The most convenient origin for our purposes is the rest position of the particle, that is, the position in which it will remain stationary when released.

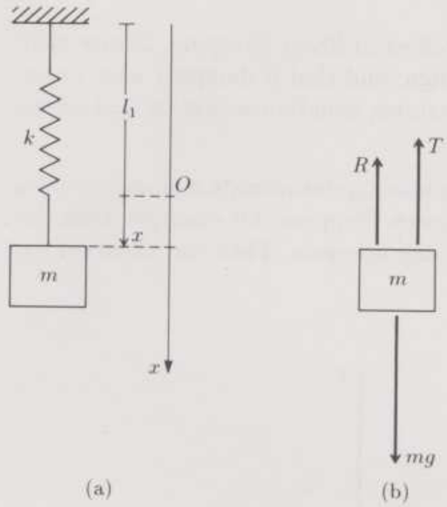


Figure 3

Exercise 1 (revision)

A particle of mass m is hung from a perfect spring of stiffness k and natural length l_0, as shown in Figure 3(a). Find the length l_1 of the spring when the particle is in equilibrium.

[*Solution on page 39*]

You tackled this problem as Exercise 9 of *Unit 7* Section 2.

As well as choosing the origin at the equilibrium position, the direction of the x-axis is taken to be downwards, as indicated in Figure 3(a). This means that:

(i) x is positive when the particle is below its equilibrium position and negative when it is above;

(ii) $\dot{x}$ is positive when the particle is moving downwards and negative when it is moving upwards;

(iii) $\ddot{x}$ is positive when the acceleration of the particle is directed downwards and negative when the acceleration is directed upwards;

(iv) the x-component of any force acting on the particle is positive if the force is directed downwards and negative if the force is directed upwards.

Figure 3(b) shows the forces which act on the particle when it is moving downwards and the spring is extended. In accordance with the convention used in *Units 4* and *7*, the arrows in Figure 3(b) indicate the directions of the forces and the quantities written beside the arrowheads are the force magnitudes. Thus T represents the magnitude of the spring force; the spring is in tension, and so it exerts an upward pull on the particle. The damping force of magnitude R also acts in the negative x-direction (since its direction is opposite to that of the velocity), and the gravitational force on the particle is downwards, as always. Hence, adding the x-components of these forces and using Newton's second law, we have

$$mg - T - R = m\ddot{x}. \qquad (1)$$

We next seek to replace the quantities T and R by expressions which involve the variables x and $\dot{x}$. The result of Exercise 1 shows that when the particle is in equilibrium (where $x = 0$) the spring has extension mg/k. If the particle has position x below its equilibrium position then the spring will be further extended by the distance x, so that its total extension for this particle position is $mg/k + x$. The magnitude of the spring force will then be the product of its stiffness and extension, that is,

$$T = k\left(\frac{mg}{k} + x\right) = mg + kx.$$

Also, since $\dot{x}$ is positive (with the particle moving downwards), the model of linear damping introduced in Subsection 1.2 gives

$$R = r|\dot{x}| = r\dot{x}.$$

Substituting these expressions for T and R into Equation (1), we obtain

$$mg - (mg + kx) - r\dot{x} = m\ddot{x},$$

or
$$m\ddot{x} + r\dot{x} + kx = 0. \tag{2}$$

This is the new equation of motion, including the effect of linear damping. Notice that all the terms on the left-hand side have the same sign, and that if damping were to be removed (by putting r equal to zero) then the remaining equation would be that of the simple harmonic model derived in *Unit 7*.

Equation (2) was derived by considering a particle moving downwards and an extended spring. However, it applies also in other circumstances. Suppose, for example, that the spring is still extended but that the particle is moving upwards. Then the situation will be as shown in Figure 4 below.

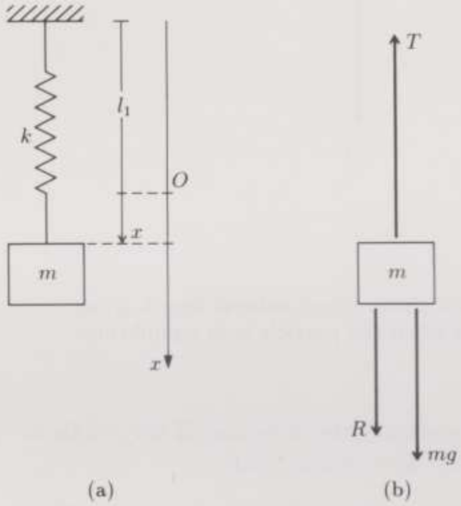


Figure 4

The direction of the spring force in Figure 4 is upwards as before, and the expression derived earlier for T in terms of x will be unchanged. The velocity $\dot{x}$ is now negative, so the damping force is directed downwards (in the positive x-direction) and has magnitude

$$R = r|\dot{x}| = -r\dot{x}.$$

Thus Equation (1) is replaced by

$$mg - T + R = m\ddot{x},$$

but when $-r\dot{x}$ is substituted for R and $mg + kx$ for T, we have

$$m\ddot{x} + r\dot{x} + kx = 0,$$

which is Equation (2) once more. This demonstrates an important feature of the linear model for the damping force: the same equation holds throughout the motion, and in deriving this equation we need consider only one particular configuration. Usually it is simplest to assume that the velocity is positive, or equivalently, that the particle is moving in the positive x-direction. Similarly, as with the simple harmonic model in *Unit 7*, there is no loss of generality in assuming that the spring is extended. The case when the spring is compressed leads to exactly the same equation of motion.

Exercise 2

Redraw Figure 3 for the case in which x measures the displacement of the particle from its equilibrium position and the x-axis is directed *upwards*. For the force diagram, assume that the spring is extended and that the particle is moving upwards. Show that the corresponding equation of motion for the particle is Equation (2) above.

[*Solution on page 39*]

The analysis above used the equilibrium position of the particle as the origin of position, for the sake of simplicity. It is, of course, possible to use other origins, and it is sometimes preferable to measure the displacement of the particle from the fixed end of the spring, as shown in Figure 5. You saw a similar alternative for the undamped perfect-spring model in *Unit 7*.

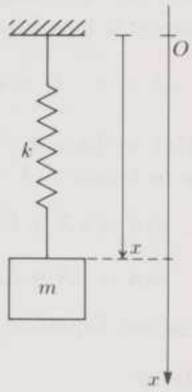


Figure 5

Example 1

A particle of mass m is hung from a fixed point by a perfect spring of stiffness k and natural length l_0. The particle is subjected to a linear damping force with damping constant r.

(i) Find the equation of motion of the particle when its displacement x is measured downwards from the fixed end of the spring.

(ii) Hence find the length l_1 of the spring when the system is in equilibrium.

Solution

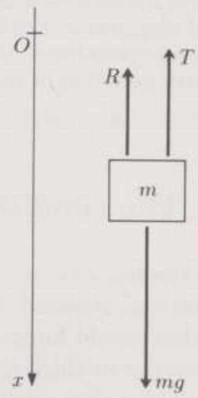


Figure 6

(i) The system is illustrated in Figure 5. We assume that the spring is extended and the particle is moving downwards, so that the velocity $\dot{x}$ is positive. (Other cases lead to exactly the same equation of motion.) The forces acting on the particle, as shown in Figure 6, are:

(a) the gravitational force, which is directed downwards and has magnitude mg;

(b) the spring force, which is directed upwards (since the spring is extended) and has magnitude $T = k(x - l_0)$;

(c) the damping force, which is directed upwards (since the particle is moving downwards) and has magnitude $R = r\dot{x}$.

Newton's second law gives

$$\begin{aligned} m\ddot{x} &= mg - T - R \\ &= mg - k(x - l_0) - r\dot{x}, \end{aligned}$$

or $$m\ddot{x} + r\dot{x} + kx = mg + kl_0,$$

which is the required equation of motion.

(ii) When the particle is in equilibrium, we have $\dot{x} = 0$ and $\ddot{x} = 0$. Substituting these into the equation of motion above produces

$$kx = mg + kl_0,$$

or $$x = \frac{mg}{k} + l_0.$$

Hence the equilibrium length of the spring is $l_1 = l_0 + mg/k$. □

Example 1(i) shows that when the particle displacement is measured from the fixed end of the spring, the corresponding equation of motion is

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0. \tag{3}$$

Notice that the left-hand side of this equation is identical in form to that of Equation (2). However, Equation (2) is homogeneous whereas Equation (3) has a non-zero right-hand side. A particular solution of the inhomogeneous linear differential equation (3) is given by

$$kx_p = mg + kl_0,$$

or $$x_p = \frac{mg}{k} + l_0,$$

which corresponds to the equilibrium position of the particle. This confirms that Equations (2) and (3) are alternative descriptions of the same physical behaviour.

Indeed, if the equation of motion is known with respect to one choice of origin, then we can find the equation of motion with respect to any other such choice by a change of variable. For example, the equation of motion of our damped vibrating system when the displacement of the particle is measured from the fixed end of the spring is given by

Equation (3). The displacement of the particle from its equilibrium position, measured downwards, is (see Figure 7)

$$X = x - l_1 = x - \left(l_0 + \frac{mg}{k}\right).$$

From this we have $x = X + l_0 + mg/k$, $\dot{x} = \dot{X}$ and $\ddot{x} = \ddot{X}$. Equation (3) can then be written in terms of X as

$$m\ddot{X} + r\dot{X} + k\left(X + l_0 + \frac{mg}{k}\right) = mg + kl_0,$$

or $$m\ddot{X} + r\dot{X} + kX = 0,$$

which is just Equation (2) with the variable X in place of x.

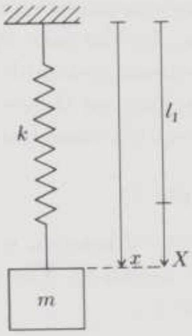


Figure 7

Exercise 3

The equation of motion (Equation (2)) for the damped vibrating system considered above is

$$m\ddot{x} + r\dot{x} + kx = 0$$

when the displacement is measured downwards from the equilibrium position. Let X be the upward displacement of the particle from floor level (see Figure 8). Assuming that the equilibrium position of the particle is a distance h above the floor, use a change of variable to derive the equation of motion in terms of X.

[*Solution on page 39*]

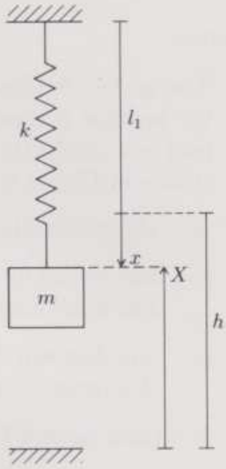


Figure 8

1.4 The condition for oscillation

When the bag of coins considered earlier in this section is displaced from its equilibrium position and released, it oscillates with a progressive decrease in amplitude. Imagine now what would happen if, instead of being suspended in air, the bag were suspended in treacle or in thick oil. Intuitively, you can see that its motion would be much more sluggish and that vibration might not occur at all. The fact that the system may or may not oscillate, depending on the amount of damping present, is reflected in the types of solution which are obtainable from the equation of motion.

The general form of this equation for the model we are considering is

$$m\ddot{x} + r\dot{x} + kx = kx_e, \tag{4}$$

where x is the position of the particle relative to some origin and x_e is the displacement of the equilibrium position from that origin. A mechanical system whose equation of motion is of this form is called a **damped linear harmonic oscillator**, or **damped harmonic oscillator** for short.

In *Unit 6* you saw that the general solution of a differential equation such as Equation (4) is the sum of any particular solution and the complementary function, which is the general solution of the associated homogeneous equation

$$m\ddot{x} + r\dot{x} + kx = 0. \tag{5}$$

A particular solution of Equation (4) is $x = x_e$, corresponding to the situation in which the particle stays at rest in its equilibrium position. To find the condition under which the particle may oscillate about this point, it is necessary to solve Equation (5). Indeed if x_e is taken to be zero, that is, if the origin for x is chosen to be at the equilibrium position of the particle, then Equation (5) is itself the equation of motion for the system. For simplicity we shall assume for the moment that this is the case, but note that the general solution of Equation (4) may be obtained directly from that of Equation (5) by addition of the term x_e.

The auxiliary equation corresponding to Equation (5) is

Here we are applying Procedure 1.1 of *Unit 6*. Note that this procedure leads to an oscillatory solution only in case (iii), when the auxiliary equation has complex roots.

$$m\lambda^2 + r\lambda + k = 0,$$

whose solutions are given by

$$\lambda = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}. \tag{6}$$

If the solution for x in terms of time t is to be oscillatory then the roots λ must be complex, for which the condition is $r^2 < 4mk$. In other words, if the damping constant

r is small enough (for example, in air) then the system can vibrate, but if r is too large (for example, in treacle) then no oscillation occurs.

If $r^2 < 4mk$ then Equation (6) can be written as

$$\lambda = \frac{-r \pm i\sqrt{4mk - r^2}}{2m}.$$

The complementary function is therefore

$$x(t) = Ae^{-rt/(2m)} \cos\left(\frac{\sqrt{4mk - r^2}}{2m}t + \phi\right)$$
$$= Ae^{-\rho t}\cos(\Omega t + \phi), \qquad (7)$$

where

$$\rho = \frac{r}{2m}, \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m}, \qquad (8)$$

and A, ϕ are constants (with A non-negative and ϕ between $-\pi$ and π) whose values may be chosen to fit the initial conditions of the system. Equations (7) and (8) give the general solution of Equation (5) provided that $r^2 < 4mk$. Equation (7) represents an oscillatory motion with **angular frequency** Ω and a diminishing amplitude $Ae^{-\rho t}$. The general shape of its graph is shown in Figure 9, which resembles in all essential respects the observed motion shown in Figure 2. Thus the inclusion of damping in the model for the motion of a bag of coins has enabled us to predict an important feature of the motion which did not show up in the simple harmonic model. The revised model is therefore an improvement on the first one.

According to Procedure 1.1 of *Unit 6*, the right-hand side of Equation (7) would have been written as
$e^{-\rho t}(B\cos\Omega t + C\sin\Omega t)$,
where ρ, Ω are as in Equations (8) and B, C are arbitrary constants. However, we have written the term in brackets here in the alternative form given by Equation (5) of *Unit 7* Section 2.

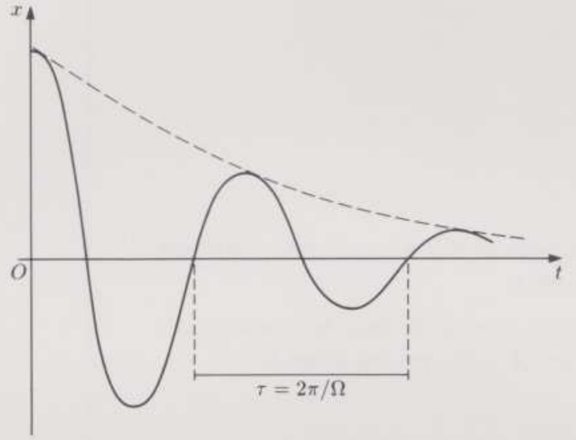


Figure 9

The **period** of the damped motion is $\tau = 2\pi/\Omega$, with Ω given by Equations (8). The corresponding period for *undamped* motion was found in *Unit 7* Section 2 to be $2\pi/\omega_0$, where $\omega_0 = \sqrt{k/m}$ is the angular frequency of simple harmonic motion. Not surprisingly, the expression (8) for Ω reduces to $\sqrt{k/m}$ when the damping constant r is equated to zero.

In the remainder of this unit we use ω_0 to denote the angular frequency of simple harmonic motion. The symbol ω, which was employed for this purpose in *Unit 7*, will be given a different meaning in Section 2.

Exercise 4

Show that, for a given mass m and spring stiffness k, the period of linearly damped oscillations is greater than the period of undamped oscillations.

[*Solution on page 39*]

The amplitude of the oscillations described by Equation (7) decreases steadily due to the exponential factor $e^{-\rho t}$, and the greater the damping (that is, the larger the value of r), the faster this amplitude decreases. To verify this, we shall look at the ratio

$$\frac{x(t+\tau)}{x(t)},$$

where $\tau = 2\pi/\Omega$ is the period of damped oscillations. Using Equation (7), we have

$$\begin{aligned} x(t+\tau) &= Ae^{-\rho(t+\tau)}\cos\left(\Omega(t+\tau)+\phi\right) \\ &= Ae^{-\rho(t+\tau)}\cos(\Omega t + 2\pi + \phi). \end{aligned}$$

Now $\cos(\Omega t + 2\pi + \phi) = \cos(\Omega t + \phi)$, and so

$$\frac{x(t+\tau)}{x(t)} = \frac{Ae^{-\rho(t+\tau)}}{Ae^{-\rho t}} = e^{-\rho(t+\tau-t)} = e^{-\rho\tau}. \tag{9}$$

As $\rho > 0$ and $\tau > 0$, we have

$$e^{-\rho\tau} < 1 \quad \text{and} \quad x(t+\tau) < x(t).$$

Equation (9) gives $e^{-\rho\tau}$ as the constant ratio of any two non-zero values of x one period apart; it applies, for example, to two consecutive maximum displacements on the same side of the rest position. As the damping constant r increases, it can be seen from Equations (8) that ρ increases while Ω decreases. Hence τ increases and $e^{-\rho\tau}$ decreases, confirming that increased damping produces a faster decrease in amplitude. The box below summarizes the results of this subsection.

Linearly damped vibrations

The equation of motion of a *damped harmonic oscillator*, namely

$$m\ddot{x} + r\dot{x} + kx = 0,$$

has an oscillating solution provided that

$$r^2 < 4mk.$$

The general solution in this case is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi),$$

where

$$\rho = \frac{r}{2m}, \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m},$$

and A, ϕ are arbitrary constants. All solutions of this form may be obtained with A non-negative and ϕ between $-\pi$ and π.

The *angular frequency* of these damped vibrations is Ω, and their *period* is $\tau = 2\pi/\Omega$. The ratio of two non-zero values of x one period apart is

$$\frac{x(t+\tau)}{x(t)} = e^{-\rho\tau}.$$

We next return for a final look at the experimental results obtained from the suspended bag of coins, showing how measurements of the amplitude can be used to estimate a value for the damping constant r.

With 20 coins in the bag, it was found that the maximum displacement of the bag from its equilibrium position decreased by a factor of about 10 over 8 complete cycles. Taking x to represent displacement from the equilibrium position, this means that

$$\frac{x(t+8\tau)}{x(t)} \simeq 0.1.$$

Now the left-hand side of this equation can be re-expressed as

$$\frac{x(t+8\tau)}{x(t+7\tau)} \times \frac{x(t+7\tau)}{x(t+6\tau)} \times \cdots \times \frac{x(t+\tau)}{x(t)} = (e^{-\rho\tau})^8 = e^{-8\rho\tau}.$$

Thus we have $e^{-8\rho\tau} \simeq 0.1$, from which

$$\rho\tau \simeq -\tfrac{1}{8}\log_e 0.1 = 0.288.$$

Using Equations (8) and $\tau = 2\pi/\Omega$ gives

$$\frac{r}{2m} \times \frac{2\pi \times 2m}{\sqrt{4mk - r^2}} \simeq 0.288,$$

or $$\frac{r^2}{4mk - r^2} \simeq \frac{0.288^2}{4\pi^2}.$$

This equation may be solved to obtain a value for r provided that numerical values for the constants m and k are available. In *Unit* 7 Subsection 2.4 it was stated that each coin has a mass of 0.01 kg, so 20 coins have a mass of 0.2 kg. It was shown also that the elastic string can be modelled by a perfect spring of stiffness $2.28\,\mathrm{N\,m^{-1}}$. Solution of the equation above with $m = 0.2$ and $k = 2.28$ results in the estimate

$$r \simeq 0.062\,\mathrm{N\,m^{-1}\,s}$$

for the damping constant.

As you saw above, the condition for oscillation is $r^2 < 4mk$. For the current case, this means that oscillation is possible provided that

$$r^2 < 4 \times 0.2 \times 2.28, \quad \text{or} \quad r < 1.35.$$

The estimate 0.062 for r is less than $\frac{1}{20}$ of this upper bound, showing that quite a small damping force is enough to make the amplitude decrease fairly rapidly. The effect of small damping on the angular frequency and hence on the period is less marked, as you are asked to demonstrate in the next exercise.

Exercise 5

Estimate the angular frequency of oscillations when there are 20 coins in the bag, as predicted by

(i) the model for linearly damped motion, using the value of r derived above;

(ii) the model for undamped motion developed in *Unit 7*.

Exercise 6

If as before $m = 0.2$, $k = 2.28$ and $r = 0.062$, find the values for A and ϕ in Equation (7) which correspond to the initial conditions $x(0) = -0.05$ and $\dot{x}(0) = 0$. (Recall that A should be non-negative and that ϕ lies between $-\pi$ and π.)

Exercise 7

Figure 10 shows a highly simplified model of a car. The particle of mass m represents the car body. This is supported by a perfect spring of stiffness k and a 'damper', which together represent the suspension. The contribution of the wheels is neglected and only the vertical motion is modelled. The damper contains a piston which moves through oil inside the cylinder. This causes a resistance to the motion of the piston which is approximately proportional to its velocity relative to the cylinder, the constant of proportionality being r.

As part of a quick test of the suspension by a potential buyer when the car is stationary on a horizontal road, the body is depressed 0.1 m below its rest position and then released. If $m = 800$, $k = 5 \times 10^4$ and $r = 9 \times 10^3$, all in the appropriate SI units, establish whether the ensuing motion is oscillatory and, if so, find the distance of the car body below its rest position after one complete cycle.

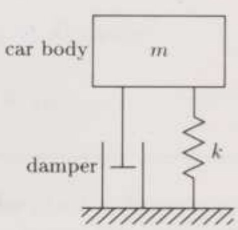


Figure 10

[*Solutions on page 39*]

1.5 Weak, critical and strong damping

In Subsection 1.3 we derived the equation of motion

$$m\ddot{x} + r\dot{x} + kx = 0$$

for a mechanical system consisting of a particle of mass m moving under the influence of a perfect spring of stiffness k and linear damping with damping constant r, where the displacement x of the particle is measured from its equilibrium position. In Subsection 1.4 the general solution of this equation was found for the case in which the inequality $r^2 < 4mk$ is satisfied. There are, of course, solutions also for the cases when $r^2 = 4mk$ and $r^2 > 4mk$, and you are asked to derive these in the next exercise.

Exercise 8

Find the general solution of the differential equation

$$m\ddot{x} + r\dot{x} + kx = 0,$$

where m, r and k are positive constants, when

(i) $r^2 < 4mk$,

(ii) $r^2 = 4mk$,

(iii) $r^2 > 4mk$.

Part (i) of this exercise is revision of Subsection 1.4.

[*Solution on page 40*]

The results of this exercise are summarized below.

Weak, critical and strong damping

The equation of motion

$$m\ddot{x} + r\dot{x} + kx = 0$$

has the following solution.

(i) If $r^2 < 4mk$ (*weak damping*) then the general solution is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi),$$

where

$$\rho = \frac{r}{2m} \quad \text{and} \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m}.$$

Here A and ϕ are constants; all solutions of this form may be obtained with A non-negative and ϕ between $-\pi$ and π.

(ii) If $r^2 = 4mk$ (*critical damping*) then the general solution is

$$x(t) = Be^{-\rho t} + Cte^{-\rho t},$$

where $\rho = r/2m$ and B, C are arbitrary constants.

(iii) If $r^2 > 4mk$ (*strong damping*) then the general solution is

$$x(t) = Be^{-\rho_1 t} + Ce^{-\rho_2 t},$$

where B, C are arbitrary constants and

$$\rho_1 = \frac{r - \sqrt{r^2 - 4mk}}{2m}, \quad \rho_2 = \frac{r + \sqrt{r^2 - 4mk}}{2m}.$$

The case $r^2 < 4mk$, which is called **weak damping**, was discussed in the previous subsection. The general solution

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi)$$

is the product of an exponential decay and a sinusoidal oscillation of angular frequency Ω. So the particle oscillates backwards and forwards about its equilibrium position with period $\tau = 2\pi/\Omega$, much as it would when there is no damping, but with the amplitude of the oscillations decaying exponentially. The graphs of typical undamped and weakly-damped oscillations are shown in Figure 11 below.

When $r^2 > 4mk$, which is called **strong damping**, the solution is

$$x(t) = Be^{-\rho_1 t} + Ce^{-\rho_2 t},$$

where ρ_1 and ρ_2 are positive constants. This solution is the sum of two decaying exponential terms. So from any initial conditions the particle returns to its equilibrium position without oscillation; the damping force dominates, and prevents any vibration. A graph for the strongly-damped case appears in Figure 11 below.

In the case $r^2 = 4mk$, which is called **critical damping**, the solution is

$$x(t) = Be^{-\rho t} + Cte^{-\rho t},$$

where ρ is a positive constant. Both parts of this solution represent a decaying motion and so the motion again fails to be oscillatory, as shown in Figure 11. Given any initial displacement and velocity, the particle returns asymptotically to its equilibrium position just as for strong damping. As suggested by the figure, critical damping gives the *fastest* return to the equilibrium position. Hence many physical devices, such as recoil mechanisms for large artillery guns, mechanical suspensions, door-closing mechanisms and weighing scales, are designed to have critical or near-critical damping.

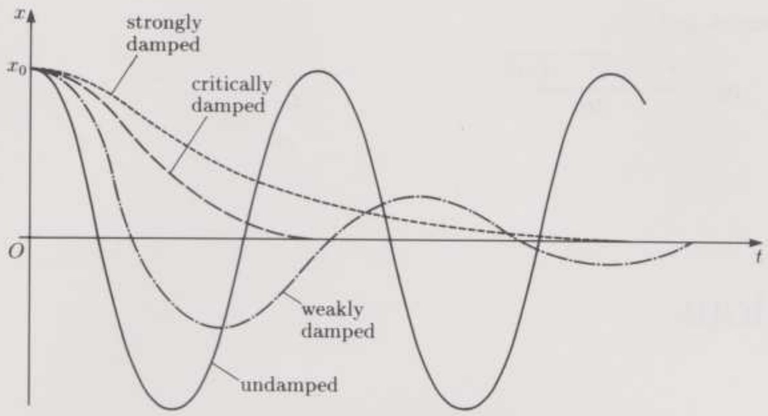


Figure 11

Exercise 9

A gun has a barrel, of mass 10^3 kg, which starts to recoil at a speed of $30\,\mathrm{m\,s^{-1}}$ when the gun is held horizontally and fired. Immediately after firing, the barrel starts to compress a spring which brings the barrel instantaneously to rest after it has recoiled a distance of 2 m. In this phase of the barrel's motion no damping is present. A linear damping mechanism is then engaged to ensure that the barrel returns to its initial position without overshooting the mark.

(i) What is the spring stiffness?

(ii) What damping constant is required for critical damping?

[*Solution on page 40*]

Summary of Section 1

1. **Damping** can be modelled by a damping force of magnitude $r|\dot{x}|$, where r is a positive constant and $\dot{x}$ is the velocity of the particle, and direction opposed to the velocity of the particle. The SI units of the damping constant are $\mathrm{N\,m^{-1}\,s}$ (or equivalently $\mathrm{kg\,s^{-1}}$).

2. The equation of motion of a particle of mass m moving under the influence of a perfect spring of stiffness k and linear damping with damping constant r is

$$m\ddot{x} + r\dot{x} + kx = 0,$$

where x is the displacement of the particle from its equilibrium position. Such a system is called a **damped harmonic oscillator**.

3. For **weak damping** ($r^2 < 4mk$) the general solution of the equation of motion is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi),$$

where

$$\rho = \frac{r}{2m} \quad \text{and} \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m}.$$

All solutions of this form may be obtained with the arbitrary constants A and ϕ restricted so that A is non-negative and ϕ lies between $-\pi$ and π. This motion is oscillatory with period $\tau = 2\pi/\Omega$. The amplitude of the oscillations decreases exponentially, so that

$$x(t+\tau)/x(t) = e^{-\rho\tau}.$$

4. For **critical damping** ($r^2 = 4mk$) the general solution is

$$x(t) = Be^{-\rho t} + Cte^{-\rho t},$$

where $\rho = r/(2m)$ and B, C are arbitrary constants. This case gives the fastest return to the equilibrium position.

5. For **strong damping** ($r^2 > 4mk$) the general solution is

$$x(t) = Be^{-\rho_1 t} + Ce^{-\rho_2 t},$$

where B, C are arbitrary constants and

$$\rho_1 = \frac{r - \sqrt{r^2 - 4mk}}{2m}, \quad \rho_2 = \frac{r + \sqrt{r^2 - 4mk}}{2m}.$$

2 Forced vibrations

2.1 Introduction

So far in this unit we have looked at the behaviour of a damped harmonic oscillator which moves freely after being started off from an initial position and with an initial velocity. We now go on to consider the motion of such a system when it is maintained in a state of vibration by some external physical agency. This 'agency' is modelled by a force changing continuously with time. The model of a car considered in Exercise 7 of Section 1 suggests one example: if the engine is started with the car stationary then it exerts an alternating force on its supports and hence on the car body, which vibrates for as long as the engine is running. As another example, think of a voltmeter which is used to measure a voltage varying with time. As the voltage increases and decreases, the pointer of the instrument will swing backwards and forwards. In both of these cases, the external force applied to the vibrating mass may vary with time in quite a complicated manner. We shall assume that the forcing agent is periodic and in that case, as will be shown later in this course (*Unit 31*), the force can be expressed as the sum of a number of sinusoidal terms. It follows that if we know the effect of a sinusoidal force we can, at least in principle, cope with any periodic variation by superimposing the effects of as many sinusoidal components as necessary. In this section, therefore, we shall concentrate on the motion of the simple damped model when it is subjected to a sinusoidal force.

This 'superposition of effects' is dealt with by applying the superposition principle of *Unit 6* Subsection 2.4 to solve the differential equation involved.

2.2 The equation of motion

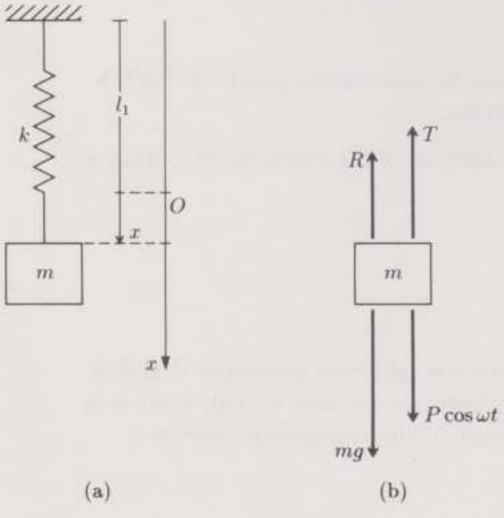


Figure 1

Figure 1(a) shows the model system of Figure 3 of Section 1, where the origin for position x is chosen to be at the particle's equilibrium position and the direction of the x-axis is taken to be downwards. This system is as before acted upon by the force due to gravity, a perfect spring and linear damping, but now there is in addition an applied sinusoidal force with x-component $P\cos\omega t$. The positive constant P is the amplitude (maximum magnitude) of the applied force, and ω is its angular frequency. These four forces are shown in Figure 1(b) for the case in which the spring is extended, the particle is moving downwards and $P\cos\omega t$ is positive. However, the same equation of motion results also for the other possible configurations of the system.

Note our notation for the three types of angular frequency occurring in this unit: (i) ω_0 for simple harmonic motion (undamped and unforced); (ii) Ω for damped harmonic motion (unforced); (iii) ω for an applied force (and for the steady-state oscillation which it causes).

As before, the magnitudes of the spring and damping forces for the case shown are respectively $T = mg + kx$ and $R = r\dot{x}$. Application of Newton's second law to Figure 1 therefore gives the equation of motion

$$mg + P\cos\omega t - r\dot{x} - (mg + kx) = m\ddot{x},$$

or

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t. \qquad (1)$$

As you may recall from *Unit 6*, the general solution of this equation is the sum of two parts: the complementary function, which is the general solution of the associated homogeneous equation (with a zero right-hand side), and a particular solution of the original equation, which depends upon the form of this equation's right-hand side. The complementary function for Equation (1) was found in Section 1; in this section we concentrate on the particular solution.

See the box on page 14.

When, as here, the equation models a physical system (with m, r and k positive), the amplitude of the complementary function decreases steadily and eventually becomes negligibly small. For this reason the complementary function is often called the **transient** part of the solution of Equation (1). On the other hand, the particular solution is persistent since, as you will see, it has a constant amplitude. It therefore remains in evidence long after the transient part of the solution has become negligible, which explains why the particular solution is known as the **steady-state** part of the solution.

This behaviour was illustrated in Figure 11 of Section 1.

Since the right-hand side of Equation (1) is sinusoidal, the easiest way to find its steady-state solution is to use the phasor method. This is based upon Euler's formula,

See Procedure 2.2(a) in *Unit 6*.

Unit 5 Subsection 4.1

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Putting ωt in place of θ gives

$$e^{i\omega t} = \cos\omega t + i\sin\omega t,$$

so that $\cos\omega t$ is the real part of $e^{i\omega t}$. It follows that $P\cos\omega t = \mathrm{Re}(Pe^{i\omega t})$. A particular solution of Equation (1) is now obtained by trying a solution of the form $x = B\cos\omega t + C\sin\omega t$, or equivalently,

$$x = A\cos(\omega t + \phi).$$

This is sinusoidal, of the same frequency as the external force but out of phase with it by an angle ϕ. Application of Euler's formula with $\omega t + \phi$ in place of θ leads to

$$x = \mathrm{Re}(Ae^{i(\omega t+\phi)}) = \mathrm{Re}(Ae^{i\phi}e^{i\omega t}).$$

We next put $Ae^{i\phi} = Z$, where Z is a complex constant (the *phasor* of the sinusoidal solution). The modulus of Z is A, the amplitude of the steady-state solution. The argument of Z is ϕ, the phase of the steady-state solution. For x and its first two derivatives we then have

$$x = \mathrm{Re}(Ze^{i\omega t}), \qquad (2)$$
$$\dot{x} = \mathrm{Re}(i\omega Ze^{i\omega t}),$$
$$\ddot{x} = \mathrm{Re}(i^2\omega^2 Ze^{i\omega t}) = \mathrm{Re}(-\omega^2 Ze^{i\omega t}).$$

Substitution of these expressions into Equation (1) produces

$$\mathrm{Re}[(-m\omega^2 + ir\omega + k)Ze^{i\omega t}] = \mathrm{Re}(Pe^{i\omega t}).$$

Hence Equation (2) will represent a solution of Equation (1) if

$$(-m\omega^2 + ir\omega + k)Z = P,$$

giving

$$Z = \frac{P}{-m\omega^2 + ir\omega + k} = \frac{P}{(k - m\omega^2) + ir\omega}$$

or $$Z = \frac{P[(k - m\omega^2) - ir\omega]}{(k - m\omega^2)^2 + r^2\omega^2}.$$

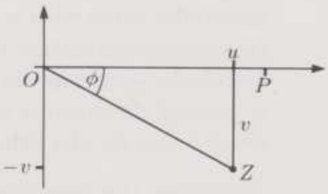


Figure 2

This is the phasor of the steady-state solution for x. Its relationship to P (the phasor of the applied force) is shown graphically in Figure 2. It may be expressed as $Z = u - iv$, where

$$u = \frac{P(k - m\omega^2)}{(k - m\omega^2)^2 + r^2\omega^2} \quad \text{and} \quad v = \frac{Pr\omega}{(k - m\omega^2)^2 + r^2\omega^2}.$$

Hence the modulus and argument of Z are respectively

$$A = |Z| = \sqrt{u^2 + v^2} = \frac{P}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}} \tag{3}$$

and $$\phi = \operatorname{Arg}(Z) = -\arccos\left(\frac{u}{|Z|}\right) = -\arccos\left(\frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}\right) \tag{4}$$

The steady-state solution of Equation (1) is therefore

$$x = A\cos(\omega t + \phi), \tag{5}$$

where A and ϕ are given by Equations (3) and (4) respectively. This sinusoidal solution, with amplitude A, is the 'output' from the system corresponding to the 'input' supplied by the forcing term with amplitude P. The angular frequency ω of the output is the same as that of the input, but between input and output there is a time delay of $-\phi/\omega$, as illustrated in Figure 3 below.

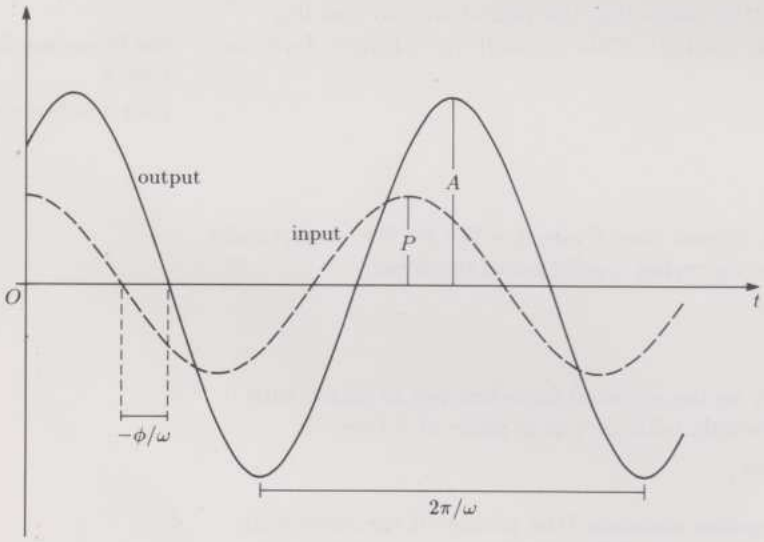


Figure 3

Exercise 1

Derive Equations (3)–(5) without the use of phasors, by substituting the trial solution

$$x = B\cos\omega t + C\sin\omega t$$

into Equation (1). (This is the alternative method of Procedure 2.2 in *Unit 6*.)

Exercise 2

Consider once more the model car described in Exercise 7 of Section 1, for which $m = 800$, $k = 5 \times 10^4$ and $r = 9 \times 10^3$, all in the appropriate SI units. While the car remains stationary on a horizontal road, its running engine produces a force on the car body which may be considered as sinusoidal, with amplitude 800 N and frequency 2.5 Hz. (Recall from *Unit 7* Subsection 2.3 that 1 Hz = 1 cycle per second.)

(i) What is the corresponding value of the angular frequency ω in rad s^{-1}?

(ii) What is the amplitude of the steady-state vibration experienced by the car body?

(iii) What is the time lag between the maximum value of the engine force being exerted in the upwards direction and the car body next reaching its maximum displacement in that direction?

Exercise 3

The equation of motion of a mechanical system is

$$2\ddot{x} + 5\dot{x} + 14x = \cos t.$$

Use the phasor method to find the steady-state oscillations of this system in the form

$$x(t) = A\cos(\omega t + \phi).$$

[*Solutions on page 40*]

You have seen above how the phasor method can be used to find the steady-state vibrations of a model system when the external force has x-component $P\cos\omega t$. The method employs the relationship

$$P\cos\omega t = \mathrm{Re}(Pe^{i\omega t}).$$

A similar phasor approach can be adopted when the applied force is of the more general form $P\cos(\omega t + \varepsilon)$, or of the equivalent form $M\cos\omega t + N\sin\omega t$. In these cases we use the equations

$$P\cos(\omega t + \varepsilon) = \mathrm{Re}(Pe^{i\varepsilon}e^{i\omega t}),$$
$$M\cos\omega t + N\sin\omega t = \mathrm{Re}\left((M - iN)e^{i\omega t}\right)$$

respectively. You are asked to consider the second of these possibilities in the exercise below.

Exercise 4

(i) Verify that

$$\mathrm{Re}\left((M - iN)e^{i\omega t}\right) = M\cos\omega t + N\sin\omega t.$$

(ii) Use the phasor method to find the steady-state solution of the equation of motion

$$\ddot{x} + 2\dot{x} + 5x = 2\cos 2t + 9\sin 2t,$$

giving your answer in the form

$$x(t) = M\cos\omega t + N\sin\omega t.$$

What are the amplitude and phase of this vibration?

[*Solution on page 41*]

2.3 The damping ratio

In the previous subsection we used the phasor method to show that the equation of motion of the forced and damped harmonic oscillator,

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t, \tag{1}$$

has the steady-state solution

$$x = A\cos(\omega t + \phi), \tag{5}$$

where

$$A = \frac{P}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}} \tag{6}$$

and $$\phi = -\arccos\left(\frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}\right). \tag{7}$$

The applied force here is completely described by two parameters, its amplitude P and angular frequency ω. For a particular set of values of m, r and k, the steady-state solution $\tilde{x}$ is a function of both P and ω, and we wish now to examine the nature of this functional dependence. This amounts to investigating how the output parameters A and ϕ depend upon the input parameters P and ω. As has already been noted, the angular frequency ω of the output is identical to that of the input.

Notice first that the phase angle ϕ is independent of the input amplitude P while, for a fixed value of the input frequency ω, the output amplitude A is proportional to P. It is not so straightforward to see the effect upon A and ϕ of varying either the angular frequency ω or the amount of damping present, but this is the principal aim of the current subsection.

We proceed by reorganizing the information contained in Equations (6) and (7), so that the quantities m, r and k are replaced by combinations which permit a clearer view of how A and ϕ react to changes in ω. In order to provide some motivation for the choice of these 'combinations' of m, r and k, we return first to Equation (1). Dividing this equation by m gives

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = \frac{P}{m}\cos\omega t. \tag{8}$$

The coefficient of x is now k/m, a quantity encountered previously. In *Unit 7* Subsection 2.3 you saw that the angular frequency of the unforced and undamped perfect-spring system was $\omega_0 = \sqrt{k/m}$, so we may replace k/m by ω_0^2. In this context we shall refer to ω_0 as the **undamped angular frequency**.

As mentioned before, we use ω_0 in this unit to denote the natural (unforced, undamped) angular frequency, rather than ω as in *Unit 7*, since currently ω represents the angular frequency of the forcing term.

The next step is to rewrite the damping term. You saw in Section 1 that the complementary function of the equation of motion is oscillatory if and only if the damping constant r is less than an upper limit given by $r^2 = 4mk$, or $r = 2\sqrt{mk}$. This leads us to define the **damping ratio** α of the system by

$$\alpha = \frac{r}{2\sqrt{mk}}.$$

The damping constant r provides an absolute value for the damping force per unit speed exerted on the particle. On the other hand, the damping ratio α gives a measure of the importance of damping *relative* to the mass m and spring stiffness k of the system. Note that α is a dimensionless quantity, since the dimensions of r ($\mathrm{N\,m^{-1}\,s}$, or $\mathrm{kg\,s^{-1}}$) are identical to those of $\sqrt{mk}$.

From the way in which α is defined, and from the results of Subsection 1.5, it follows that a system moving in accordance with the complementary function of Equation (8) is

(i) weakly damped (oscillatory) if $\alpha < 1$,

(ii) critically damped if $\alpha = 1$,

(iii) strongly damped if $\alpha > 1$.

Figure 4 shows graphs of the complementary function for a particular system for four different values of α; the initial conditions are the same for all of them. This is essentially a repetition of Figure 11 in Section 1.

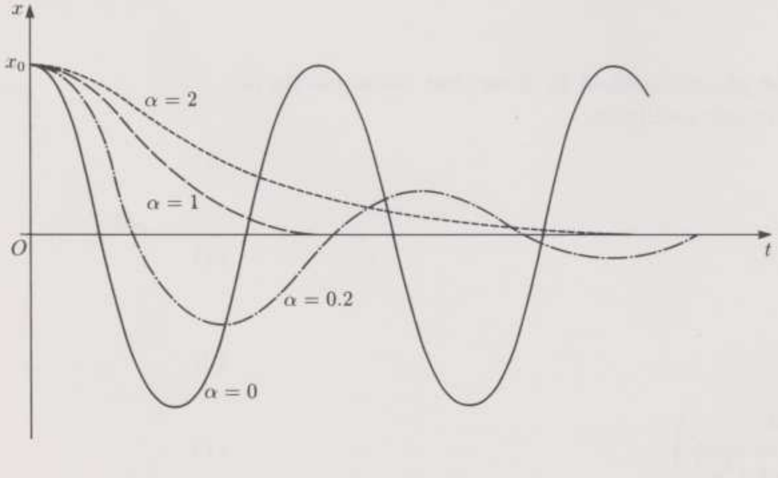


Figure 4

From the equations

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \alpha = \frac{r}{2\sqrt{mk}}, \tag{9}$$

we have

$$\frac{r}{m} = \frac{2\alpha\sqrt{mk}}{m} = 2\alpha\sqrt{\frac{k}{m}} = 2\alpha\omega_0,$$

so that Equation (8) can now be written as

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t. \tag{10}$$

Exercise 5

Determine the phasor for the steady-state solution of Equation (10). Hence find expressions for the amplitude and phase angle of this motion.

Exercise 6

A linearly damped perfect-spring system with sinusoidal applied force has the equation of motion

$$4\ddot{x} + 9\dot{x} + 100x = 9\cos 6t.$$

(i) Find the values of the undamped angular frequency and the damping ratio.

(ii) Determine the amplitude, phase and angular frequency of the steady-state solution.

[*Solutions on page 41*]

In Exercise 5 you showed that the steady-state solution of Equation (10) is $x = A\cos(\omega t + \phi)$, where

$$A = \frac{P/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}} \tag{11}$$

and $$\phi = -\arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}\right). \tag{12}$$

These expressions could also have been obtained directly from Equations (6) and (7), using Equations (9) to replace k by $m\omega_0^2$ and r by $2\alpha m\omega_0$.

The forced and damped harmonic oscillator

The equation of motion

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t$$

of the forced and damped harmonic oscillator can be rewritten as

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t,$$

in terms of the *undamped angular frequency*

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and the *damping ratio*

$$\alpha = \frac{r}{2\sqrt{mk}}.$$

The steady-state solution $x = A\cos(\omega t + \phi)$ of this system has amplitude

$$A = \frac{P/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}$$

and phase

$$\phi = -\arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}\right).$$

The best way to show the significance of Equations (11) and (12) is to draw graphs. In the case of Equation (12), what we want to demonstrate is the effect on the value of ϕ of varying ω and α. But the equation also contains ω_0, so one possibility would be to plot, for each of a range of values of ω_0, a set of curves of ϕ against ω, each separate curve corresponding to a particular value of α. This might mean drawing a large number of sets of curves, which would require much calculation and plotting. In order to simplify the procedure, we note that Equation (12) can be rewritten as

$$\phi = -\arccos\left(\frac{1-(\omega/\omega_0)^2}{\sqrt{(1-(\omega/\omega_0)^2)^2+4\alpha^2(\omega/\omega_0)^2}}\right)$$

which shows that ϕ can be regarded as a function of the two variables α and ω/ω_0. We therefore plot graphs of ϕ against ω/ω_0 for particular values of α, as shown in Figure 5(b) below. This enables all of the required information to be presented by drawing just one set of curves.

The same sort of approach can be applied to Equation (11), except that the task of representing graphically the variation of A with ω and α is further complicated here by the presence of m and P as well as ω_0. Noting that

$$\frac{P}{m} = \frac{P}{k}\frac{k}{m} = \omega_0^2\frac{P}{k},$$

and putting $M = Ak/P$, Equation (11) can be rewritten as

$$M = \frac{1}{\sqrt{(1-(\omega/\omega_0)^2)^2+4\alpha^2(\omega/\omega_0)^2}}, \tag{13}$$

where, as before, the right-hand side is a function of the two variables α and ω/ω_0. The resulting graphs of M against ω/ω_0 for various values of α are shown in Figure 5(a) below.

Notice that M, like α and ω/ω_0, is a dimensionless quantity. For a given spring stiffness k, the ratio of the output displacement amplitude A to the input force amplitude P can be found from the value of M.

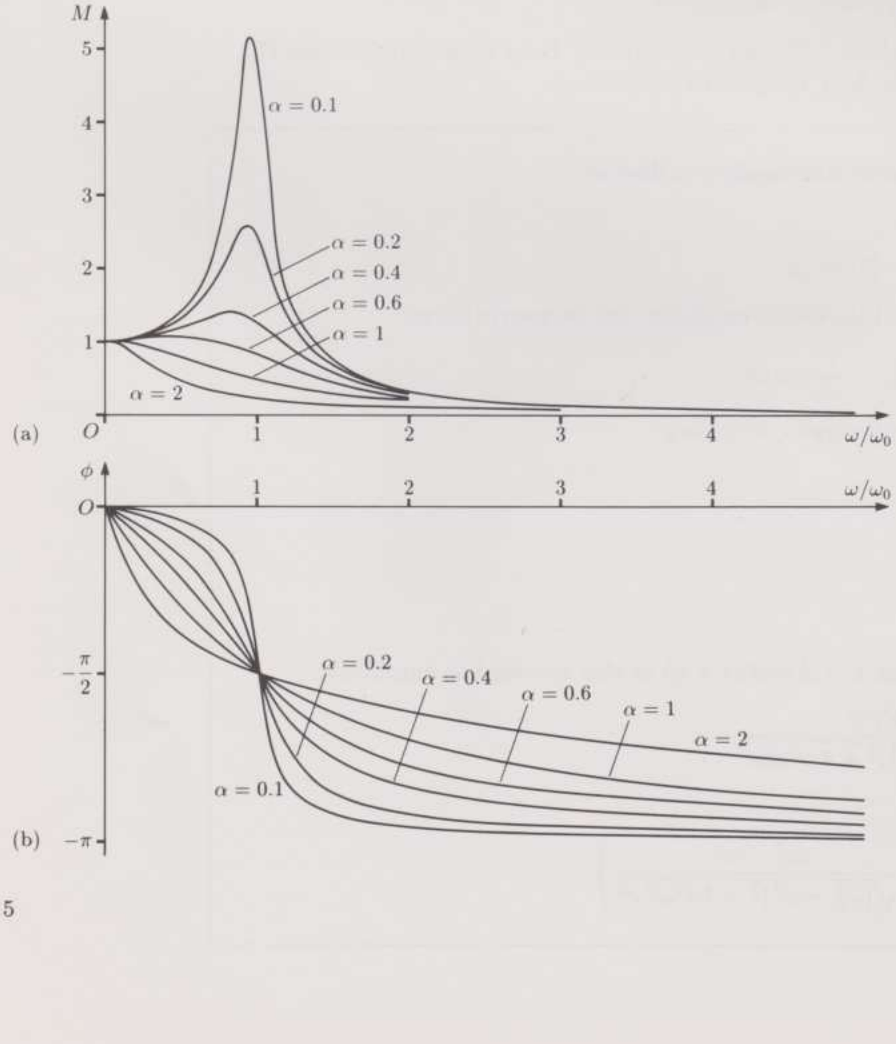


Figure 5

Exercise 7

Calculate the value of M which corresponds to the steady-state solution of the differential equation given in Exercise 6 above.

[*Solution on page 42*]

2.4 Resonance

The most noticeable feature of the graph of M against ω/ω_0 in Figure 5(a) is that for low values of α it shows a definite peak near $\omega/\omega_0 = 1$. This means that, when the angular frequency ω of the sinusoidal external force is close to the undamped angular frequency ω_0 of the spring–mass system, the steady-state amplitude of the displacement of the particle becomes large compared with its values for the same force amplitude over most of the frequency range. The motion of the system at these relatively large amplitudes is known as **resonance**. As α increases, the resonance peak becomes less pronounced and moves towards lower values of ω/ω_0. When $\alpha = 1/\sqrt{2}$, the peak is at $\omega/\omega_0 = 0$ (as you are asked to show in Exercise 10 below). For $\alpha \geq 1/\sqrt{2}$ the curve has negative slope for all positive values of ω/ω_0.

Resonance is found in many physical systems and may be considered desirable, as in the case of a radio receiver which is tuned to an input signal of a particular carrier frequency, so that inputs at other frequencies are neglected. However, there are other situations where resonance may be most undesirable. In the case of a car, large vertical movements of the car body will certainly be uncomfortable and may also be dangerous and destructive.

Figure 5(a) may be used to represent the response of an instrument which measures and records a physical quantity such as, for example, electrical voltage. The movement of the instrument's pointer (and attached parts of the mechanism) can be modelled by the movement of a particle, and the voltage causes, and is proportional to, the external applied force. If the voltages to be measured are periodic and have a range of different frequencies, then it is preferable to avoid re-calibrating the instrument for every different frequency; indeed, it might be the case that the voltage consists of several sinusoidal functions of different frequencies superimposed. Hence the same value of M is required for as wide a frequency range as possible. Of the curves drawn in Figure 5(a), that for $\alpha = 0.6$ meets most closely the condition that M should be approximately constant over the greatest possible frequency range. The value usually chosen in practice is $\alpha = 0.65$. In order to achieve this damping ratio, the instrument movements often incorporate deliberately added linear, or nearly linear, damping of a mechanical or electrical nature.

In order for the output to be a faithful representation of the input we need to have the same value of M for all the frequency components, as already explained, so that each component of the input makes the appropriate contribution to the output. In addition, we need to ensure that the time shift between different components (the interval between the instants at which different components reach their maxima, say) is not affected by passing through the instrument. If it were, the output would be a distorted version of the input. So, to avoid distortion, one wants the time lag introduced by the instrument to be the same for each component. This time lag equals $-\phi/\omega$ (see Figure 3, page 18), so that we want the phase ϕ to be directly proportional to the input frequency ω. As it happens, for $\alpha = 0.65$ this is almost true over a frequency range similar to that for which M is approximately constant. Figure 5(b) illustrates this.

Figure 5(b) also shows that, for all values of α and ω/ω_0, the phase ϕ lies between $-\pi$ and 0, and that $\phi = -\pi/2$ when $\omega = \omega_0$.

Exercise 8

A voltmeter is modelled by a forced and damped harmonic oscillator with damping ratio $\alpha = 0.6$ and undamped angular frequency $\omega_0 = 20\,\mathrm{rad\,s^{-1}}$. Estimate from Figure 5(a) the frequency range over which the instrument will reproduce the amplitude of a sinusoidal input voltage with a relative error of less than 10%.

Exercise 9

Use the graphs in Figure 5(a) to check your answers to parts (ii) and (iii) of Exercise 2 (page 19).

Exercise 10

Write the right-hand side of Equation (13) in terms of α and $p = \omega/\omega_0$. Using this expression show that, for a given value of α, the maximum value of M occurs

(i) when $p = \sqrt{1-2\alpha^2}$, if $\alpha < 1/\sqrt{2}$;

(ii) when $p = 0$, if $\alpha \geq 1/\sqrt{2}$.

[*Solutions on page 42*]

Summary of Section 2

1. The equation of motion of a damped harmonic oscillator subject to an applied sinusoidal force of amplitude P and angular frequency ω can be written as

$$m\ddot{x} + k\dot{x} + rx = P\cos\omega t.$$

The steady-state **forced vibrations** of this system can be found by using the phasor method.

2. This equation of motion can be rewritten in the form

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t,$$

where the **undamped angular frequency** ω_0 is defined (as in *Unit 7*) by

$$\omega_0 = \sqrt{\frac{k}{m}},$$

and the **damping ratio** α is defined by

$$\alpha = \frac{r}{2\sqrt{mk}}.$$

3. The steady-state vibration of the forced and damped harmonic oscillator is given by

$$x = A\cos(\omega t + \phi),$$

where

$$A = \frac{P/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}$$

and $$\phi = -\arccos\left(\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}\right).$$

4. For $\alpha < 1/\sqrt{2}$, the amplitude of the forced vibrations exhibit a maximum at a particular value of the forcing angular frequency ω. This effect is known as **resonance**.

3 The perfect dashpot

3.1 Introducing the dashpot

In Section 1 we augmented the *Unit 7* model of a particle–spring system by the inclusion of linear damping, and in Section 2 an external force acting upon the particle was added. This section shows how the linear model for damping can be applied to configurations of particle, spring and damping which differ from those encountered previously. These configurations arise from situations in which an external force acts upon a point of the system other than the particle itself.

For example, consider the vertical oscillations experienced by a motor car as a result of its motion along an undulating road. This may be modelled by considering the car to be at rest with the road surface moving beneath it, the effect of the undulations being represented by a sinusoidal displacement of the wheels about the mean road level. In predicting the vibrations of the car body under these circumstances we need to include a linear damping force which is proportional to the *relative* velocity between the body and wheels, rather than a force proportional to the absolute vertical velocity as in the earlier model for a stationary car. Also, we now have to incorporate an external force described by the vertical displacement of the wheels rather than a force acting directly upon the car body.

The main purpose of this section is to enable you to write down the correct equation of motion for the particle when the system consists of any configuration involving a perfect spring, linear damping and an applied force. As a first step, Figure 1 shows a diagrammatic means of indicating linear damping. It represents a piston inside a cylinder, or *dashpot*. One form of dashpot is used as the shock absorber on a car (see Exercise 7 in Section 1), where the cylinder is full of oil and the relative motion of the piston forces the oil to flow through the annular space between piston and cylinder. This produces a resistance to the relative motion. For modelling purposes we assume that damping is produced by a **perfect dashpot**, for which the resistance is proportional to the *relative velocity* of the dashpot's ends (or equivalently, to the rate of change of the dashpot's length). The constant of proportionality is denoted as before by r, but is now referred to as the **dashpot constant** rather than the damping constant. If the instantaneous length of the perfect dashpot is l then the magnitude of the damping force produced is $r|\dot{l}|$.

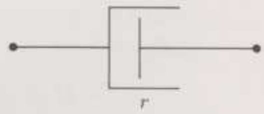


Figure 1

The occurrence of a perfect dashpot on a diagram indicates the presence of linear damping, irrespective of whether the damping being modelled is produced by an actual dashpot or by some other physical means. In this respect the role of the perfect dashpot is analogous to that of the perfect spring, which indicates the presence of a force depending linearly on the relative displacement of two end-points, whether or not the real force being modelled is applied by the agency of a spring.

Figure 2 shows how we could draw a perfect dashpot alongside a perfect spring to represent the model of Section 1 for the suspended bag of coins. The damping effects here are due not to a physical dashpot but to air resistance and internal friction in the rubber string. Since the point of suspension is fixed, the damping force magnitude $r|\dot{l}|$ in this instance is equal to $r|\dot{x}|$, where x is the displacement of the particle from any fixed point. However, this will not always be the case.

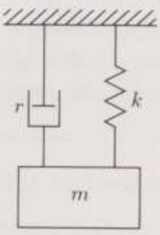


Figure 2

Figure 3 shows a perfect dashpot with both ends in motion. The positions of these ends with respect to some fixed point are denoted by x and y, so that their velocities are respectively $\dot{x}$ and $\dot{y}$. Since the dashpot length is $l = x - y$, the magnitude of the damping force exerted by the dashpot is $r|\dot{l}| = r|\dot{x} - \dot{y}|$. Now if $\dot{x} > \dot{y}$, that is, if the dashpot is lengthening, then the force exerted on whatever is connected to the dashpot at either end is directed towards the dashpot. (Imagine that you are pulling the two ends of the dashpot apart with your hands. The dashpot will resist by exerting forces trying to stop your hands moving apart. The force on your right hand is to the left, and the force on your left hand is to the right.) By the same token, if $\dot{x} < \dot{y}$, that is, if the dashpot is shortening, then the forces on objects attached to the ends will be directed away from the dashpot, again resisting any change to the length of the dashpot.

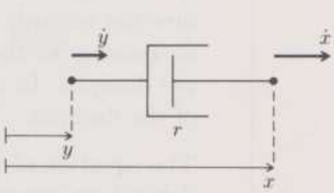


Figure 3

Example 1

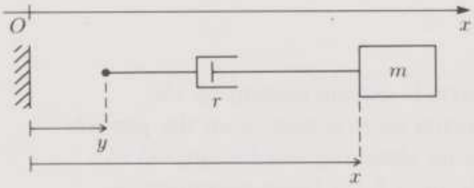


Figure 4

A particle is attached to one end of a perfect dashpot, as shown in Figure 4. The displacements of the particle and of the other end of the dashpot from a fixed point O are denoted respectively by x and y, where $x > y$. Find the x-component of the dashpot force acting on the particle when

(i) the length of the dashpot is increasing;

(ii) the length of the dashpot is decreasing.

Solution

The length of the dashpot is $l = x - y$, and so $\dot{l} = \dot{x} - \dot{y}$.

(i) When the length of the dashpot is *increasing*, $\dot{l}$ is positive and so $|\dot{l}| = \dot{l} = \dot{x} - \dot{y}$. The magnitude of the dashpot force is therefore $R = r|\dot{l}| = r(\dot{x} - \dot{y})$. In this case the dashpot force is directed towards the dashpot, as shown in Figure 5, trying to resist the increase in the length of the dashpot. Hence the x-component of the dashpot force is

$$-R = -r(\dot{x} - \dot{y}).$$

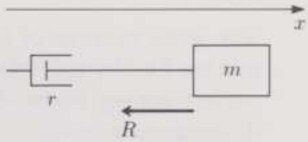


Figure 5

(ii) When the length of the dashpot is *decreasing*, $\dot{l}$ is negative and so $|\dot{l}| = -\dot{l} = -(\dot{x} - \dot{y})$. Hence the magnitude of the dashpot force is $R = r|\dot{l}| = -r(\dot{x} - \dot{y})$. Again, this force tries to resist the change in the length of the dashpot, and so is directed away from the dashpot, as indicated in Figure 6. Therefore the x-component of the dashpot force is

$$R = -r(\dot{x} - \dot{y}). \quad \square$$

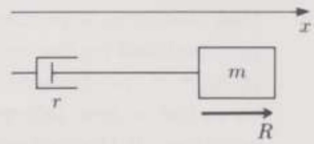


Figure 6

This example demonstrates that the x-component of the dashpot force takes the same form whether the length of the dashpot is increasing or decreasing. Consequently, the equation of motion of a particle attached to a dashpot will also have the same form in these two cases. We shall therefore usually assume, when deriving an equation of motion, that the dashpot length is increasing (so that the dashpot force is directed towards the centre of the dashpot). However, the resulting equation of motion will be valid also for the case when the length of the dashpot is decreasing.

The perfect dashpot

The perfect dashpot is a diagrammatic representation of a type of force, rather than of the mechanism which produces it. It represents a resistive force proportional to the rate of change of the dashpot's length l. In other words, the force has magnitude $R = r|\dot{l}|$, where r is a positive constant known as the *dashpot constant*.

If the dashpot's length is increasing (so that $\dot{l}$ is positive), the dashpot force is directed towards the centre of the dashpot. If the length of the dashpot is decreasing (so that $\dot{l}$ is negative), the force is directed away from the centre of the dashpot. In either case the dashpot force resists any change in the length of the dashpot.

The equation of motion for a particle attached to a perfect dashpot can be derived by assuming that the dashpot is increasing in length.

Example 2

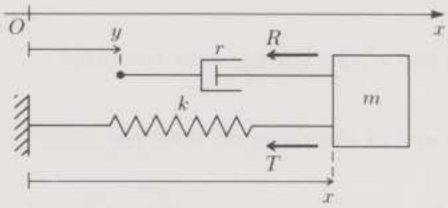


Figure 7

A particle of mass m is attached to a fixed point O by a horizontal perfect spring of natural length l_0 and stiffness k. The particle is also attached by a horizontal dashpot of dashpot constant r to a point whose displacement with respect to the fixed point O is y, as shown in Figure 7. Find the equation of motion for the particle, assuming that it is otherwise free to move in a horizontal direction.

Solution

In deriving the equation of motion we shall assume that the spring is extended and that the length of the dashpot is increasing, so that the forces act in the directions indicated in Figure 7. We shall measure the displacement x of the particle from the fixed point O. The extension of the spring is $x - l_0$, so the magnitude of the spring force is

$$T = k(x - l_0).$$

The length of the dashpot is $l = x - y$, and so $\dot{l} = \dot{x} - \dot{y}$. Now $\dot{l}$ is positive, since the length of the dashpot is increasing. Hence the magnitude of the dashpot force is

$$R = r\dot{l} = r(\dot{x} - \dot{y}).$$

By Newton's second law, the equation of motion for the particle is

$$\begin{aligned} m\ddot{x} &= -T - R \\ &= -k(x - l_0) - r(\dot{x} - \dot{y}), \end{aligned}$$

or $$m\ddot{x} + r\dot{x} + kx = kl_0 + r\dot{y}. \quad \square$$

Exercise 1

Find the equation of motion for the particle in Figure 2, taking x to be the downward displacement of the particle from the fixed ends of the spring and dashpot, and l_0 to be the natural length of the spring. Compare your answer with Equation (3) in Section 1 (page 9).

Exercise 2

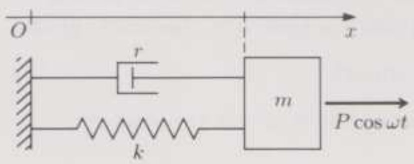


Figure 8

Find the equation of motion for the particle in Figure 8, which is attached to a perfect dashpot and perfect spring (of natural length l_0) as well as being subjected to an external force whose x-component is $P \cos \omega t$.

[*Solutions on page 42*]

Consider once more the equation of motion derived in Example 2 for the situation shown in Figure 7. Note that, as it stands, this equation of motion cannot be solved to obtain an expression for the function $x(t)$, since the $r\dot{y}$ term on the right-hand side is an unknown function of t. This situation can be altered by specifying the position function $y(t)$ of the left-hand end of the dashpot. This left-hand end is referred to as the *forcing point* of the system, since it is the point at which an external force is applied.

There are essentially three different configurations for a single-particle system, connected to one perfect spring and one perfect dashpot, which has an external force applied at a point other than the particle:

(i) the external force may be applied to the other end of the spring, with the other end of the dashpot held fixed;

(ii) the external force may be applied to the other end of the dashpot, with the other end of the spring held fixed;

(iii) the other ends of the spring and dashpot may be joined together, and the external force applied to this joint end.

Example 2 looked at an instance of case (ii). We next consider an occurrence of case (iii), including specification of the forcing point motion. Provided that this motion is sinusoidal, the mathematics required to solve the equation of motion for the particle is much the same as that employed earlier.

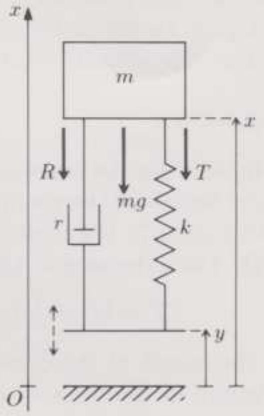


Figure 9

Figure 9 is a diagram of the highly simplified model of a car which was mentioned at the beginning of this section. This model is obtained by neglecting the mass of the wheels and concentrating the wheel suspensions into one pair of perfect spring and perfect dashpot elements. The spring and the dashpot are assumed to have the same length at all times, with their lower ends being at the same horizontal level. The 'input' $y(t)$ is a sinusoidal displacement of the common lower end of the spring and dashpot. This input represents the undulations of the road, which the car is traversing at a certain speed. The 'output' $x(t)$ is the vertical displacement of the particle of mass m which represents the car body. Both x and y are taken to be positive upwards and measured from a fixed horizontal level, which is the mean height of the road surface.

Assuming that the spring is extended and that the dashpot is increasing in length, the directions of the forces on the car body are as shown in Figure 9. The extension of the spring is $x - y - l_0$, and the rate of increase of the dashpot's length is $\dot{x} - \dot{y}$. So the magnitudes of the spring and dashpot forces are respectively

$$T = k(x - y - l_0) \quad \text{and} \quad R = r(\dot{x} - \dot{y}).$$

The gravitational force, of magnitude mg, acts vertically downwards. The equation of motion of the car body is therefore

$$\begin{aligned} m\ddot{x} &= -mg - T - R \\ &= -mg - k(x - y - l_0) - r(\dot{x} - \dot{y}), \end{aligned}$$

or

$$m\ddot{x} + r\dot{x} + kx = kl_0 - mg + ky + r\dot{y}. \tag{1}$$

If the undulations of the road are sinusoidal about its mean level and the car travels at constant speed, then the forcing point motion will also be sinusoidal. If these forcing point oscillations have amplitude A_0 and angular frequency ω then $y = A_0 \cos \omega t$, giving

$$m\ddot{x} + r\dot{x} + kx = kl_0 - mg + A_0(k \cos \omega t - r\omega \sin \omega t). \tag{2}$$

This equation for forced and damped motion is similar to those which you met in Section 2. Its steady-state solution has the form

$$x(t) = l_0 - mg/k + A \cos(\omega t + \phi),$$

where $l_0 - mg/k$ is the height at which the particle could remain in equilibrium if no external force were applied. This constant part of the solution is obtained by finding a particular solution of the equation

$$m\ddot{x} + r\dot{x} + kx = kl_0 - mg.$$

The sinusoidal part of the solution is derived by finding a particular solution of the equation

$$m\ddot{x} + r\dot{x} + kx = A_0(k \cos \omega t - r\omega \sin \omega t). \tag{3}$$

Exercise 3

Derive an expression for the phasor of the steady-state solution of Equation (3), and hence derive expressions for the amplitude A and phase angle ϕ of the forced vibrations described by Equation (2).

[*Solution on page 42*]

Under certain circumstances the position x of the particle relative to some fixed point may be of less interest than its position z relative to the forcing point. Since $z = x - y$, it is a simple matter to obtain the appropriate differential equation for z directly from that for x. For example, the height of the car body above the road surface in the model of Figure 9 is $z = x - y$. On replacing x by $z + y$ in Equation (1), we obtain

$$m\ddot{z} + r\dot{z} + kz = kl_0 - mg - m\ddot{y}.$$

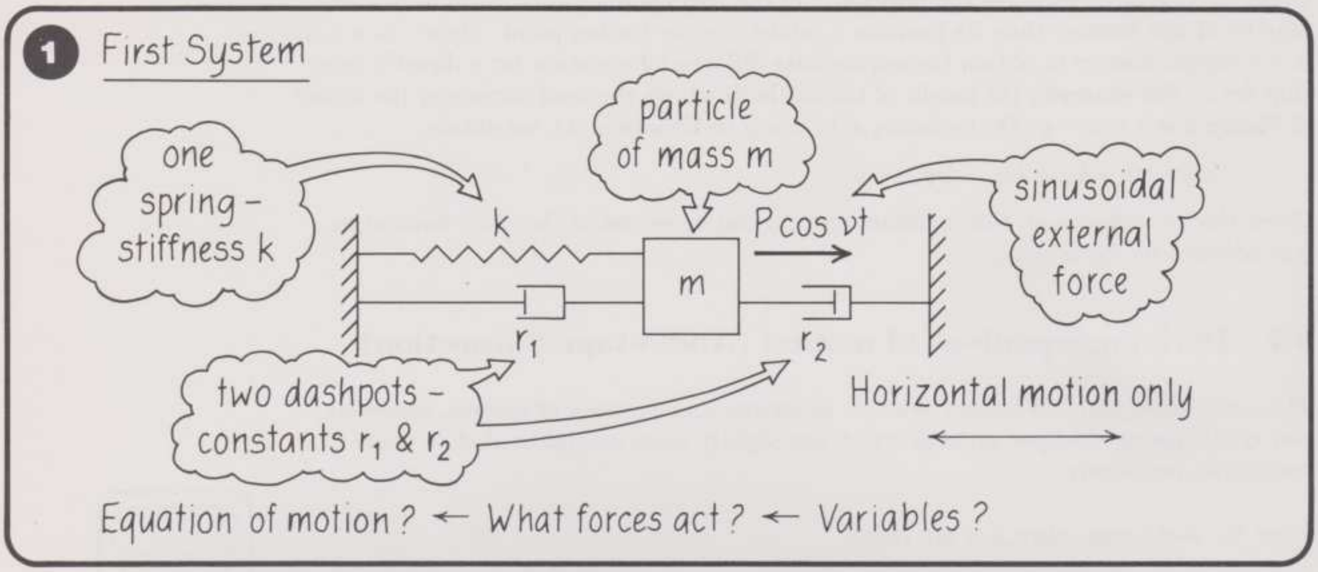
Given that $y = A_0 \cos \omega t$, this equation for $z(t)$ can be solved in the same manner as was adopted for Equation (2).

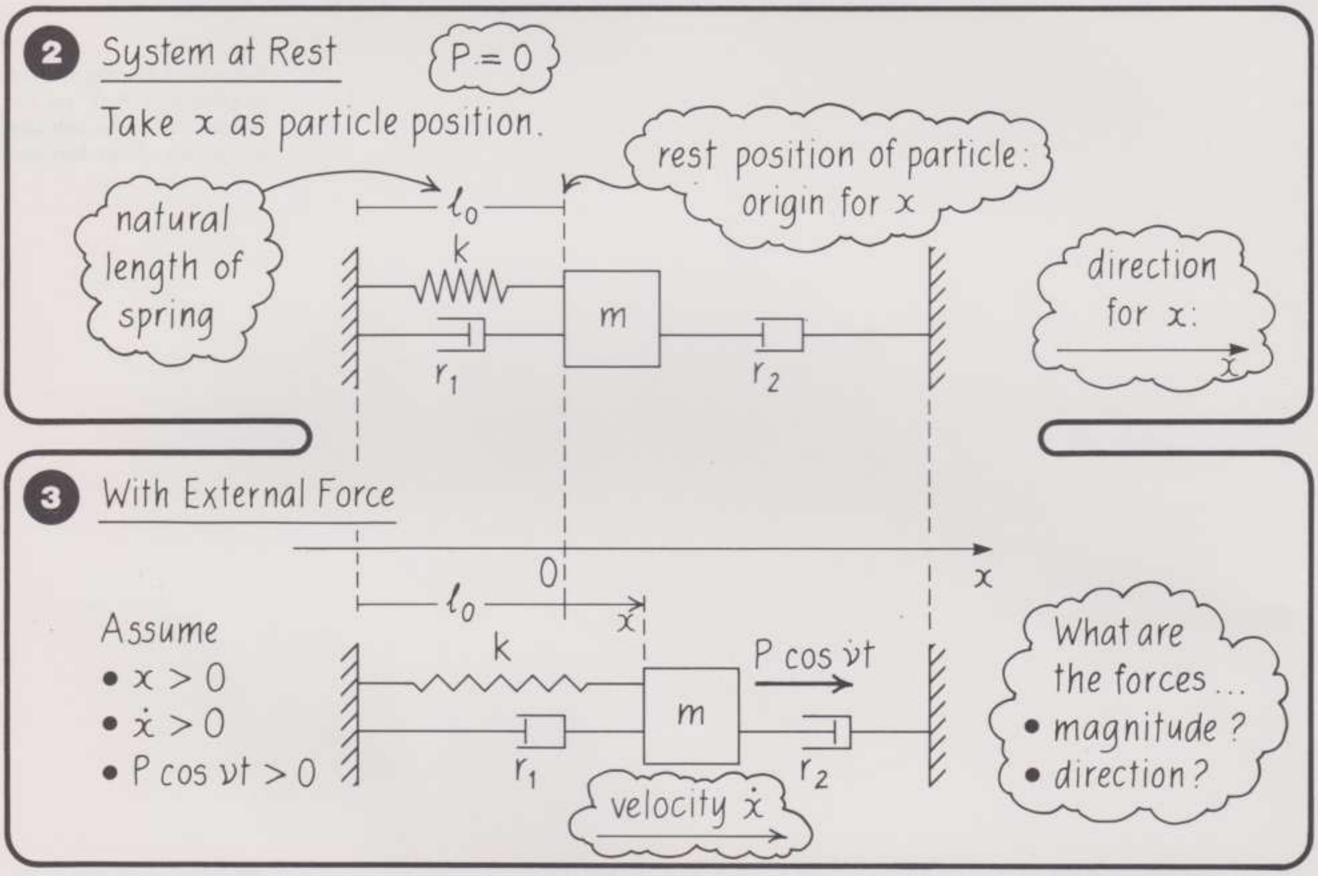
3.2 Deriving equations of motion (Audio-tape Subsection)

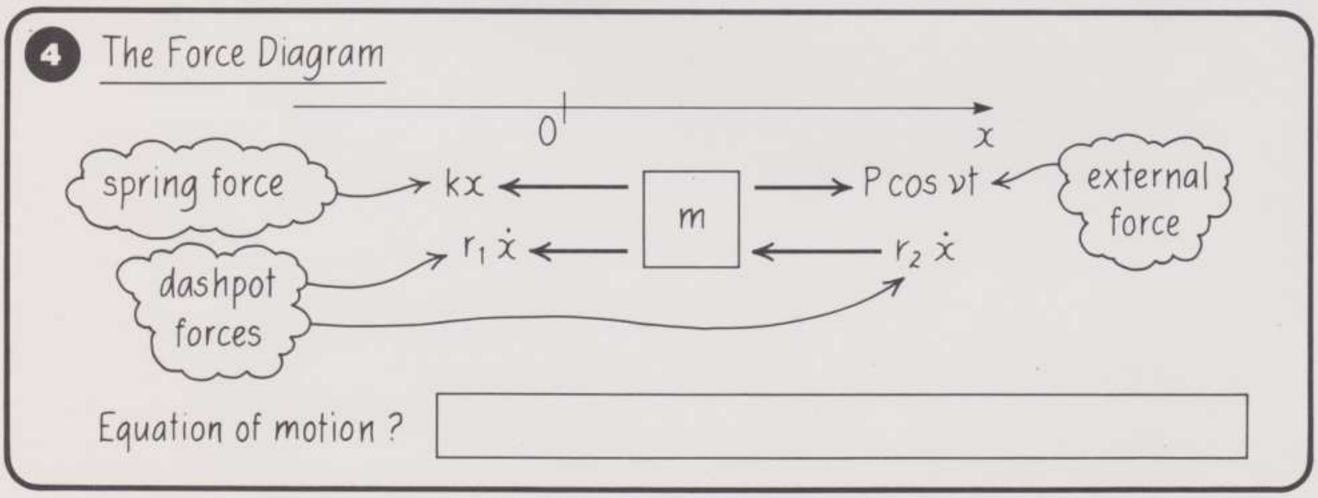
The audio-tape provides further practice in setting up equations of motion, analysing two model spring–dashpot systems which are slightly more complicated than those considered previously.

Start the audio-tape when you are ready.

Note that during the tape the symbol ν (Greek 'nu') is used in place of ω for the angular frequency of the forcing term.


1 First System
one spring - stiffness k
particle of mass m
sinusoidal external force
k
m
P cos νt
r₁
r₂
two dashpots - constants r₁ & r₂
Horizontal motion only
Equation of motion ? ← What forces act ? ← Variables ?


2 System at Rest
P = 0
Take x as particle position.
natural length of spring
rest position of particle: origin for x
ℓ₀
k
m
r₁
r₂
direction for x:
x
3 With External Force
0
x
ℓ₀
x
Assume
• x > 0
• ẋ > 0
• P cos νt > 0
k
m
P cos νt
r₁
r₂
velocity ẋ
What are the forces ...
• magnitude ?
• direction ?


4 The Force Diagram
0
x
spring force
kx
m
P cos νt
external force
r₁ ẋ
r₂ ẋ
dashpot forces
Equation of motion ?

5 The Equation of Motion

x-components of forces are

- external: $P\cos\nu t$
- spring: $-kx$
- dashpots: $-r_1\dot{x} - r_2\dot{x}$

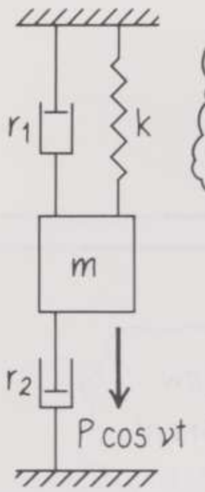
$$\boxed{m\ddot{x} = P\cos\nu t - kx - r_1\dot{x} - r_2\dot{x}}$$

Newton's 2nd law: $m\ddot{x} = x$-component of total force

Re-arrange: $m\ddot{x} + (r_1 + r_2)\dot{x} + kx = P\cos\nu t$

or $\boxed{m\ddot{x} + r\dot{x} + kx = P\cos\nu t}$ $\quad (r = r_1 + r_2)$

6 Turned through 90°

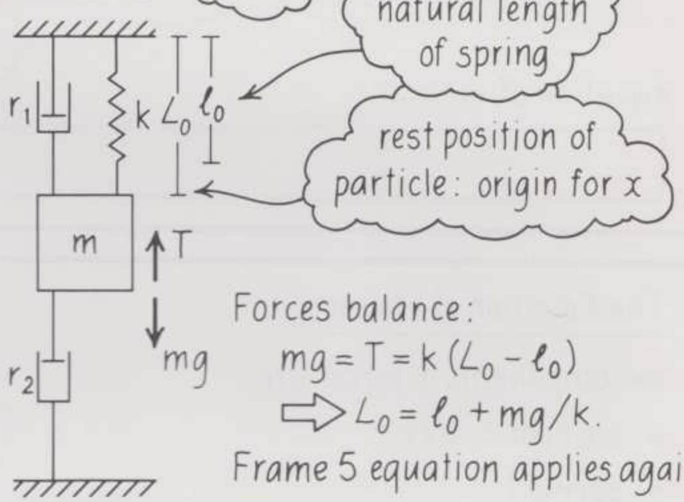


How does it differ?

gravity acts!

7 At Rest

$P = 0$



Forces balance:

$mg = T = k(L_0 - \ell_0)$

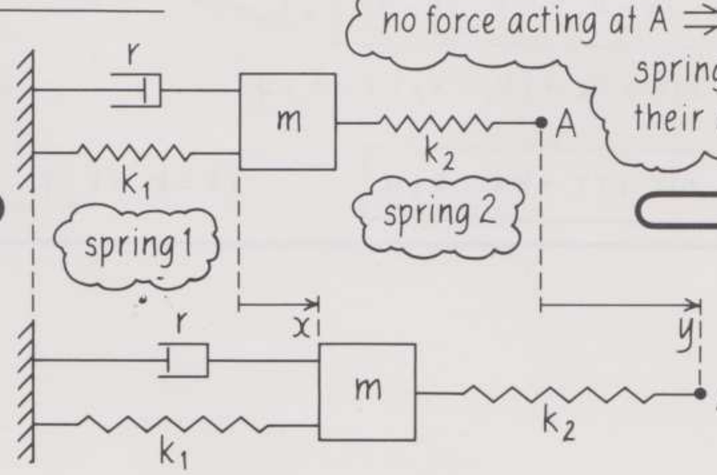
$\Rightarrow L_0 = \ell_0 + mg/k.$

Frame 5 equation applies again.

8 Second System – at Rest

two springs, one dashpot

no force acting at A $\Rightarrow$ springs are at their natural lengths



9 In Motion

Assume $x > 0$, $\dot{x} > 0$, $y > x$.

What are the force magnitudes?

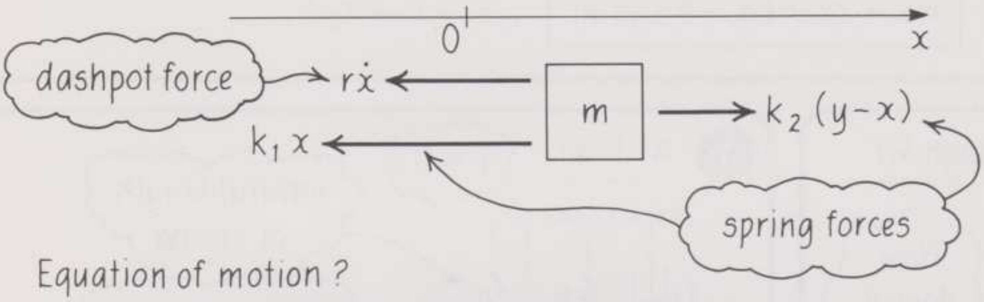
- spring 1
  extension = ☐
  tension = ☐
- spring 2
  extension = ☐
  tension = ☐
- dashpot
  length change rate = ☐
  force magnitude = ☐

10 The Force Magnitudes

- spring 1: extension = $\boxed{x}$, tension = $\boxed{k_1 x}$
- spring 2: extension = $\boxed{y-x}$, tension = $\boxed{k_2(y-x)}$
- dashpot: length change rate = $\boxed{\dot{x}}$, force magnitude = $\boxed{r\dot{x}}$

Diagram of forces?

11 The Force Diagram



Equation of motion?

$\boxed{\phantom{m\ddot{x} = -r\dot{x} - k_1 x + k_2(y-x)}}$

12 The Equation of Motion

x-components of forces are

- dashpot: $-r\dot{x}$
- spring 1: $-k_1 x$
- spring 2: $+k_2(y-x)$

Newton's 2nd law: $m\ddot{x}$ = x-component of total force

$$\boxed{m\ddot{x} = -r\dot{x} - k_1 x + k_2(y-x)}$$

Re-arrange: $m\ddot{x} + r\dot{x} + (k_1 + k_2)\,x = k_2 y$

or $\boxed{m\ddot{x} + r\dot{x} + kx = k_2 y}$ $\quad (k = k_1 + k_2)$

The remaining exercise in this section again asks you to derive an equation of motion, and illustrates how the response graphs of Figure 5 in Section 2 (page 22) can be adapted for use in different circumstances.

Exercise 4

Figure 10 shows a model of a solid body being moved vertically through water. The perfect dashpot represents the resistance of the water, and the perfect spring represents the resilience of a towing rope. (Note that a perfect spring can reasonably be used as a model for a rope only if the rope is in tension, since a rope, unlike a spring, cannot sustain a compressive force.)

(i) Initially the top end of the spring is held at a fixed position. If the natural length of the spring is l_0, what is its length when the particle is in equilibrium?

(ii) The top end of the spring is now moved along a vertical straight line, with displacement y measured upwards from its previous fixed position. The corresponding displacement x of the particle is measured upwards from its equilibrium position. By making suitable assumptions, draw the diagram of forces acting on the particle, and derive the equation of motion for the particle.

(iii) Suppose that $k = 200\,\mathrm{N\,m^{-1}}$ and $m = 4\,\mathrm{kg}$, and that the damping ratio is $\alpha = 0.2$. Suppose further that the top end of the spring is given a sinusoidal displacement about its original fixed position, of amplitude 0.05 m and frequency 2 Hz. By comparing the equation of motion derived in part (ii) with Equation (1) of Section 2 (page 17), use Figure 5(a) in Section 2 (page 22) to estimate the amplitude of the particle's steady-state oscillations.

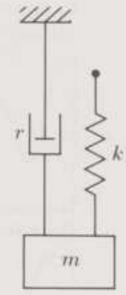


Figure 10

[*Solution on page 43*]

Summary of Section 3

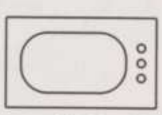
1. **A perfect dashpot** represents a resistive force which is proportional to the relative velocity between two components of a mechanical system.

2. The dashpot force has magnitude $R = r|\dot{l}|$, where l is the length of the dashpot and r is a positive constant known as the **dashpot constant**. If the dashpot's length is increasing, then the dashpot force is directed towards the centre of the dashpot; if the length is decreasing, then the force is directed away from the centre of the dashpot.

3. The equation of motion of a particle attached to a perfect dashpot can be derived by assuming that the dashpot's length is increasing. This equation of motion applies also to the case where the dashpot is contracting.

4 Off the record: resonance and damping (Television Section)

Now watch the television programme.
Read the following after viewing the programme.


TV8

Allan Solomon introduced the programme by saying that to understand why the tone-arm of a record player can misbehave, we need to understand damping and forcing in simpler mechanical systems. Thereafter the programme divided naturally into three parts, each of which considered the vibrations of a particular apparatus:

Part 1: the unforced, damped vibrations of an apparatus modelled by Figure 1;
Part 2: the forced vibrations of an apparatus modelled by Figure 2;
Part 3: the forced vertical vibrations of a tone-arm, modelled by Figure 3.

Part 1 of the programme showed how a simple mass–spring–dashpot system behaved, and introduced the model for it shown in Figure 1. David Broadhurst found that a dashpot containing air made little difference to the oscillations observed without a dashpot: the motion still resembled simple harmonic motion (see Figure 4(a)). With water in the dashpot the damping was more noticeable, though the motion was still oscillatory (Figure 4(b)). Strong damping occurred with oil in the dashpot, when there were no oscillations but just a simple decay to the equilibrium position (Figure 4(c)).

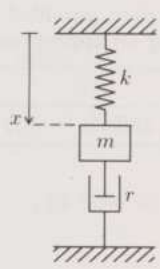


Figure 1

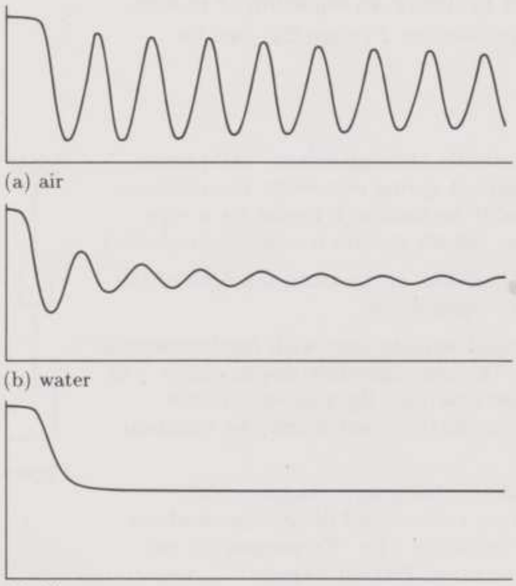


Figure 4 Output traces from a simple mass–spring–dashpot system.

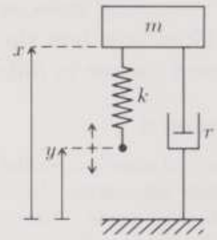


Figure 2

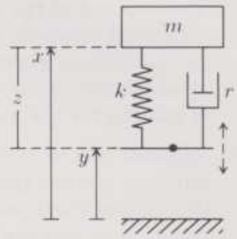


Figure 3

It was then shown that the equation of motion for the model in Figure 1 was

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0,$$

This is Equation (3) of Section 1, which you derived once more in Exercise 1 of Section 3.

where l_0 is the natural length of the spring. This equation was solved, and graphs of the predicted motion were shown for three values of the damping ratio $\alpha = r/(2\sqrt{mk})$ (see Figure 5, in which $x_0 = l_0 + mg/k$ is the equilibrium position of the particle). For $\alpha < 1$ damped oscillations were forecast, while for $\alpha > 1$ an absence of oscillations was predicted. Critical damping, with $\alpha = 1$, gave the fastest decay to the static position. The observed results were clearly in qualitative agreement with the model's predictions. One quantitative test is the subject of the exercise below.

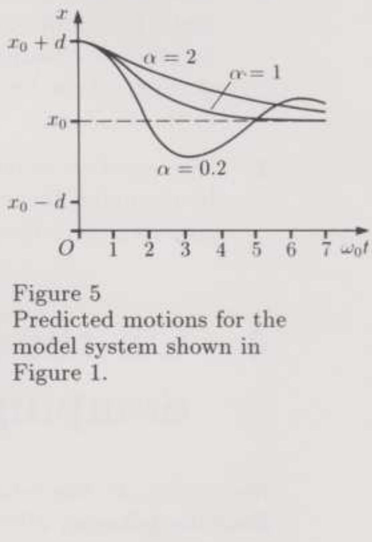


Figure 5
Predicted motions for the model system shown in Figure 1.

Exercise 1

In Subsection 1.4 it was shown that, for the model of Figure 1, the ratio of particle displacements from the equilibrium position at times one period τ apart is

$$\frac{x(t+\tau)}{x(t)} = e^{-\rho\tau},$$

where $\rho = r/(2m)$. It follows from this that

$$\frac{x(t+n\tau)}{x(t)} = e^{-n\rho\tau} \quad \text{for } n = 1, 2, \ldots .$$

Thus if $A_0, A_1, A_2, \ldots$ are the heights above the mean of successive peaks on the graph of $x(t)$, and $A'_0, A'_1, A'_2, \ldots$ are the depths below the mean of successive troughs (with A'_0 between A_0 and A_1, say), then

$$A_n = A_0 e^{-n\rho\tau} \quad \text{and} \quad A'_n = A'_0 e^{-n\rho\tau},$$

from which

$$A_n + A'_n = (A_0 + A'_0)e^{-n\rho\tau},$$

or

$$\log_e\left(\frac{A_0 + A'_0}{A_n + A'_n}\right) = n\rho\tau.$$

We add the two previous equations here because it is easier to measure $A_n + A'_n$ from the trace than it is to measure A_n or A'_n individually.

Using the data provided below for the water-filled dashpot, test this prediction of the model by drawing an appropriate graph.

n	0	1	2	3	4
$100(A_n + A'_n)$	5.1	2.2	1.3	0.8	0.6

[*Solution on page 44*]

Part 2 of the programme dealt with the forced oscillations of an apparatus modelled by Figure 2. The design of this apparatus is indicated in Figure 6.

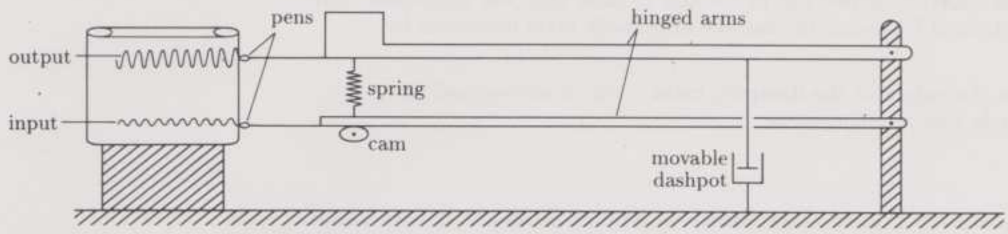


Figure 6 The apparatus used in Part 2 of the television programme.

The motor-driven cam made the bottom of the spring move up and down sinusoidally with a fixed amplitude and at a forcing frequency that could be varied. This was the input oscillation,

$$y = y_0 + A_0 \cos \omega t,$$

recorded by the lower of the two traces. The steady-state output oscillation,

$$x = x_0 + A \cos(\omega t + \phi),$$

was recorded by the upper trace. In addition to the spring connecting the two arms of the apparatus, there was a dashpot connecting the upper arm with a point at a fixed level. The geometry is such that the effective damping constant can be increased by sliding the dashpot to the left.

Initially the output was studied for three different forcing frequencies, with the same amount of damping. The first frequency gave an amplitude ratio A/A_0 greater than unity. At a higher frequency the amplitude ratio was even greater, but at a higher frequency still it decreased. Thus the system exhibited resonance, since the amplitude ratio had a maximum greater than unity (see Figure 7).

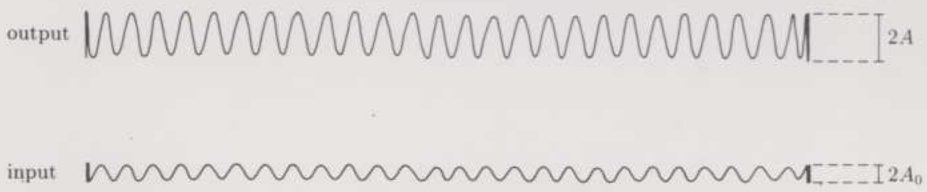


Figure 7 Input and output traces for the apparatus of Figure 6 with light damping, near the natural frequency.

The apparatus was modelled by the system of Figure 2, leading to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = kl_0 - mg + ky, \tag{1}$$

where l_0 is the natural length of the spring. With $y = y_0 + A_0 \cos \omega t$, Equation (1) becomes

$$m\ddot{x} + r\dot{x} + kx = k(l_0 + y_0 - mg/k) + kA_0 \cos \omega t. \tag{2}$$

This has steady-state solution

$$x = x_0 + A \cos(\omega t + \phi),$$

where $x_0 = l_0 + y_0 - mg/k$ and

$$\frac{A}{A_0} = \frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + (2\alpha\omega/\omega_0)^2}}. \tag{3}$$

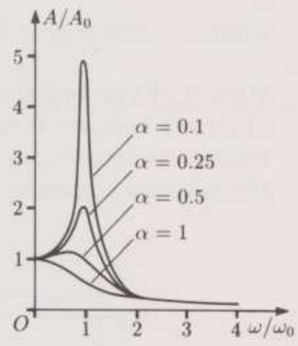


Figure 8
The predictions of Equation (3) for different values of the damping ratio α.

Graphs of the amplitude ratio from Equation (3) were provided for values of the damping ratio α between 0.1 and 1.0 (see Figure 8). These showed that with sufficiently large damping, an absence of resonance was predicted.

Exercise 2

(i) Show that the equation of motion for the system in Figure 2 is given by Equation (1).

(ii) By comparison with the analysis carried out in Section 2, show that the sinusoidal term in the steady-state solution of Equation (2) has the amplitude ratio described by Equation (3).

(iii) State a lower bound for the values of the damping ratio α which correspond to non-resonant behaviour in this model system.

[*Solution on page 44*]

Data taken from the apparatus of Figure 6 are shown in Figure 9 below. It was found that a reasonable fit to these data could be obtained with the graph for $\alpha = 0.2$ in Figure 8. The prediction of no resonance for sufficiently large damping was also tested. With the dashpot moved considerably to the left, a sweep of input frequencies revealed a steady decrease of output amplitude with increasing frequency. This indicated an absence of resonance, in qualitative agreement with the prediction.

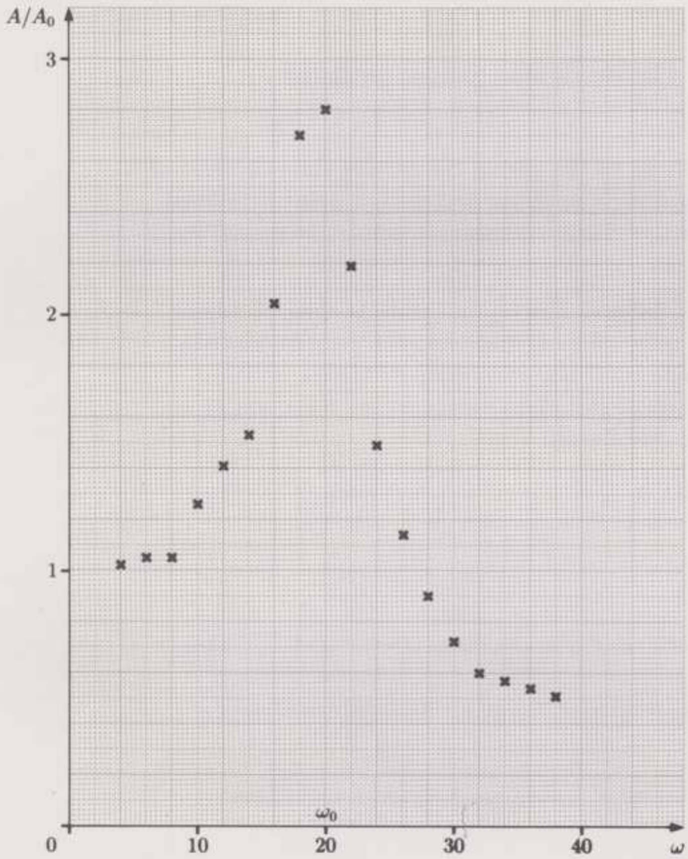


Figure 9 Amplitude ratio data obtained from the apparatus of Figure 6.

Part 3 of the programme dealt with the forced vertical oscillations of the tone-arm of a record player. It was shown that the large demonstration apparatus of Figure 10 exhibited damped vibrations when unforced and resonant vibrations when forced. Like the previous apparatus, this one involved the pivoting motion of an arm.

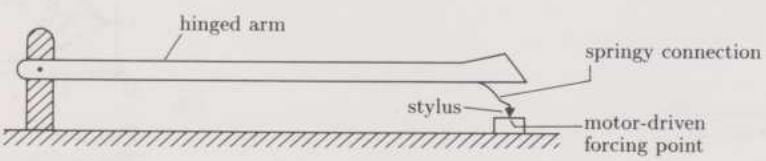


Figure 10 Demonstration apparatus for Part 3 of the programme.

The motion of the tone-arm was modelled by the system shown in Figure 3, in which the connection between the stylus and tone-arm is assumed to involve a perfect dashpot (linear damping) as well as a perfect spring. In the following exercises you are asked to find the equation of motion of this system, and the amplitude of the steady-state output when it is subjected to a sinusoidal input vibration. Note that interest centres on the motion of the tone-arm relative to the record surface, as described by the variable $z = x - y$.

Exercise 3

(i) Show that the equation of motion of the model system shown in Figure 3 is

$$m\ddot{x} + r\dot{x} + kx = kl_0 - mg + r\dot{y} + ky.$$

(ii) Hence show that the displacement $z = x - y$ of the particle relative to the forcing point satisfies the differential equation

$$m\ddot{z} + r\dot{z} + kz = kl_0 - mg - m\ddot{y}.$$

Exercise 4

If the system shown in Figure 3 is subjected to a sinusoidal input vibration $y = A_0 \cos \omega t$, show that the amplitude ratio for steady-state vibrations of the particle relative to the forcing point is given by

$$\frac{A}{A_0} = \frac{(\omega/\omega_0)^2}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}},$$

where $\omega_0 = \sqrt{k/m}$ is the undamped angular frequency and $\alpha = r/(2\sqrt{mk})$ is the damping ratio.

[*Solutions on page 44*]

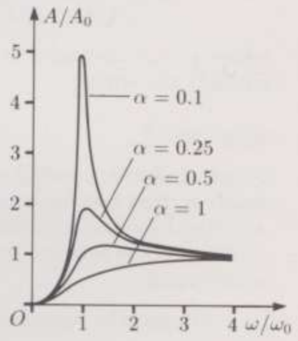


Figure 11
The predictions from the result of Exercise 4.

In the previous exercise you derived an expression which predicts the amplitude ratio for the tone-arm. Some corresponding graphs are shown in Figure 11. It was stated that the vertical oscillations of a real tone-arm can be reasonably accounted for by this model with a value of about 0.2 for α. This is true, at low frequencies, but there are a couple of remarks which need to be made. First, the cartridge detects two sorts of motion: relative vertical motion between the stylus and tone-arm, and relative horizontal motion (the existence of these two signals is what makes stereo reproduction possible). Secondly, the arm is not perfectly rigid and can therefore flex in response to the motion of the stylus (this is important at frequencies higher than those we consider). These two factors mean that the real system has more 'degrees of freedom' than have been incorporated by modelling the arm as a particle moving in one dimension.

Despite these limitations, the model has certain satisfactory features. It can be shown that it predicts resonance for $\alpha < 1/\sqrt{2}$ (using the method of Exercise 10 in Section 2, page 24), and it suggests that we should increase the damping if we want to reduce the amplitude ratio of the system near the natural frequency. However, this natural frequency is typically 60 radians per second (that is, $60/(2\pi) \simeq 10$ oscillations per second, or 10 hertz). The frequencies at which vibrations are forced by the undulations in the record grooves, producing the music that we hear, are between ten and a thousand times greater in value. This led to the question of what could be causing the record-player to misbehave so badly.

The fault turned out to be a bad warp on the record. All records are warped to some extent, causing potentially an unwanted resonant response to forcing at low frequencies. This effect can be reduced by increasing the damping. It was shown that a similar tone-arm on a second turntable did not exhibit such large forced vibrations. The difference between the tone-arms was that the second one had a specially designed miniature dashpot connecting the arm to the surface of the record. (To make the comparison a fair one, a sliding weight on the second arm had been adjusted to compensate for the additional weight of the dashpot.)

5 End of unit exercises

Exercise 1

A small heavy object is hung from the ceiling by a spring, and it is found that the spring is extended by 19.6 cm in the static position. The object is then attached to the floor by two identical dashpots, chosen to give a total effective damping ratio $\alpha = 1$.

(i) Devise a model for this system, and obtain its equation of motion, giving numerical values for the coefficients. Take the value of g to be $9.8\,\mathrm{m\,s^{-2}}$.

(ii) One of the two dashpots is now removed. What is the new equation of motion? Predict the period of the damped vibrations that can now be observed. If the object is released from rest with the spring at its natural length, what is the maximum subsequent spring extension predicted by the model?

Exercise 2

Suppose that, in addition to the forces acting on the object in Exercise 1(ii), there is an externally-produced magnetic force, acting downwards with magnitude

$$P_0 + P\cos\omega t \qquad (P_0 > P > 0).$$

Show that the resulting forced vibrations have amplitude

$$A = \frac{d}{\sqrt{(1-p^2)^2 + p^2}},$$

where $p = \omega/\omega_0$ and d is the distance by which a constant downward force of magnitude P extends the spring.

Exercise 3

Figure 1 shows a model of a seismometer, which consists of a particle of mass m suspended inside a rigid box by a perfect spring and attached also to a perfect dashpot. The box is given a vertical sinusoidal motion of angular frequency ω and amplitude A_0, and the displacement z of the particle relative to the box is recorded. This displacement is zero when the particle is in equilibrium.

(i) Obtain the equation of motion of the particle relative to the box.

(ii) Derive the steady-state solution of this equation of motion.

(iii) Show that, if the damping ratio α and the ratio ω_0/ω of undamped to forcing angular frequencies are both small, then $z \simeq -y$. (This result demonstrates that the device can be used to record the effect of an earthquake, provided that the damping is light and the undamped angular frequency is small compared with the frequencies of ground movement.)

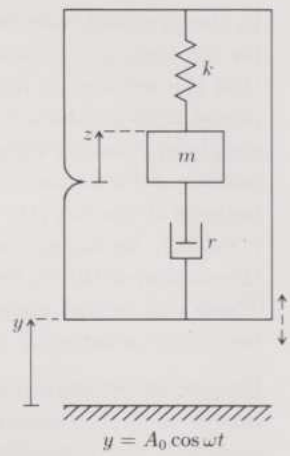


Figure 1

Exercise 4

This exercise again refers to the model of Figure 1, and to the results of Exercise 3.

(i) Show that, if the damping ratio α and the ratio ω/ω_0 of forcing to undamped angular frequencies are both small, then the relative displacement z of the particle is approximately proportional to the acceleration of the box, with the constant of proportionality being independent of the forcing frequency. (This result demonstrates that the device can be used as an accelerometer, provided that the damping is light and the undamped angular frequency is large compared with the frequencies of accelerated motion to be measured.)

(ii) If $\alpha = \omega/\omega_0 = 0.1$, calculate the ratio of the approximate amplitude of the box acceleration (as indicated by the instrument) to the true amplitude.

[*Solutions on page 45*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1.

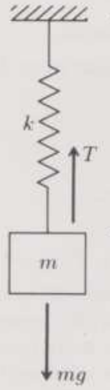


Figure 1

In the equilibrium position, the spring will be extended and so the spring force will be upwards. The magnitude of this force is $T = k(l_1 - l_0)$. The only other force on the particle is that due to gravity. As the particle is in equilibrium we have

$$T = mg,$$

which leads to

$$l_1 = l_0 + mg/k.$$

2. The figure required is shown below. The force diagram assumes, as stated in the question, that the spring is extended and the particle is moving upwards.

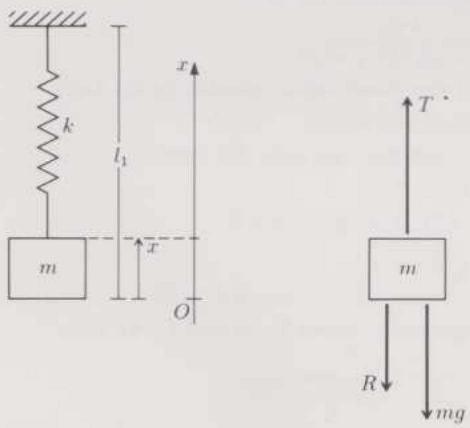


Figure 2

As in Exercise 1 we have $T = mg$ when the particle is at its equilibrium position. If the particle is below this position then, since $x < 0$, the *additional* tension in the spring is $k(-x) = -kx$. Alternatively, if the particle is above its equilibrium position (but with the spring still extended) then, since $x > 0$, the *decrease* in tension is kx. In either case, the total tension in the spring is given by

$$T = mg - kx.$$

The particle is moving upwards, so that $\dot{x} > 0$ and

$$R = r|\dot{x}| = r\dot{x}.$$

Newton's second law then gives

$$m\ddot{x} = T - R - mg$$
$$= -kx - r\dot{x}$$

or $m\ddot{x} + r\dot{x} + kx = 0,$

which is Equation (2) of Section 1 once more.

3. The particle is at a distance x below the equilibrium position, which is at a height h above the floor. So the height of the particle above the floor is $X = h - x$. Hence we have

$$x = h - X, \quad \dot{x} = -\dot{X} \quad \text{and} \quad \ddot{x} = -\ddot{X}.$$

Substituting these expressions into the equation

$$m\ddot{x} + r\dot{x} + kx = 0,$$

we obtain

$$m(-\ddot{X}) + r(-\dot{X}) + k(h - X) = 0$$

or $m\ddot{X} + r\dot{X} + kX = kh.$

4. The angular frequency Ω of damped oscillations is

$$\Omega = \frac{\sqrt{4mk - r^2}}{2m} = \sqrt{\omega_0^2 - \frac{r^2}{4m^2}},$$

where $\omega_0 = \sqrt{k/m}$ is the undamped angular frequency. Hence $\Omega^2 < \omega_0^2$, so that $\tau = 2\pi/\Omega$ is greater than the undamped period $2\pi/\omega_0$.

5. **(i)** The damped angular frequency is

$$\Omega = \frac{\sqrt{4mk - r^2}}{2m}.$$

Here $m = 0.2$, $k = 2.28$ and $r = 0.062$, so that

$$\Omega = \frac{\sqrt{4 \times 0.2 \times 2.28 - (0.062)^2}}{0.4}$$
$$\simeq \frac{1.35}{0.4} \simeq 3.37\,\text{rad s}^{-1}.$$

(ii) The undamped angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.28}{0.2}} \simeq 3.38\,\text{rad s}^{-1}.$$

6. Equation (7) of Section 1 is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi).$$

The derivative of this is

$$\dot{x}(t) = -\rho Ae^{-\rho t}\cos(\Omega t + \phi) - \Omega Ae^{-\rho t}\sin(\Omega t + \phi).$$

Using the given initial conditions, we have

$$x(0) = A\cos\phi = -0.05, \tag{1}$$
$$\dot{x}(0) = -\rho A\cos\phi - \Omega A\sin\phi = 0. \tag{2}$$

Now $\Omega \simeq 3.37$ from Exercise 5(i) and, using the first of Equations (8) in Section 1,

$$\rho = \frac{r}{2m} \simeq \frac{0.062}{0.4} = 0.155.$$

Use of Equation (1) to substitute for $A\cos\phi$ in Equation (2) therefore leads to

$$A\sin\phi = \frac{0.05\rho}{\Omega} \simeq \frac{0.05 \times 0.155}{3.37} \simeq 0.0023. \tag{3}$$

Equations (1) and (3) can be solved by the method of *Unit 7* Subsection 2.3, yielding

$$A = \sqrt{(-0.05)^2 + (0.0023)^2} \simeq 0.050,$$
$$\phi = \arccos(-0.05/A) \simeq 3.1.$$

7. The motion will be oscillatory if $r^2 < 4mk$. Here we have

$$r = 9 \times 10^3, \quad m = 800, \quad k = 5 \times 10^4,$$

so that

$$r^2 = 8.1 \times 10^7,$$
$$4mk = 4 \times 800 \times 5 \times 10^4 = 1.6 \times 10^8.$$

Hence the inequality is satisfied, and the motion is oscillatory.

The ratio of displacements one period apart is

$$\frac{x(t+\tau)}{x(t)} = e^{-\rho\tau},$$

where

$$\rho = \frac{r}{2m}, \quad \tau = \frac{2\pi}{\Omega} \quad \text{and} \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m}.$$

Therefore

$$\rho\tau = \frac{2\pi r}{\sqrt{4mk - r^2}} = \frac{2\pi \times 9 \times 10^3}{\sqrt{1.6 \times 10^8 - 8.1 \times 10^7}} \simeq \frac{5.65 \times 10^4}{8.89 \times 10^3} \simeq 6.36.$$

From this it follows that

$$e^{-\rho\tau} \simeq 1.73 \times 10^{-3}.$$

Hence one complete cycle after being released the displacement of the car body below its rest position will be

$$0.1 \times 1.73 \times 10^{-3} = 1.73 \times 10^{-4}\,\text{m},$$

or about 0.2 mm.

8. For the differential equation

$$m\ddot{x} + r\dot{x} + kx = 0,$$

the auxiliary equation is

$$m\lambda^2 + r\lambda + k = 0$$

with roots

$$\lambda = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}.$$

In each of the following three cases we apply Procedure 1.1 of *Unit 6*.

(i) $r^2 < 4mk$ (complex roots for λ)

The roots are $\lambda = -\rho \pm i\Omega$, where

$$\rho = \frac{r}{2m} \quad \text{and} \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m},$$

so the general solution of the differential equation is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi).$$

Here A and ϕ are arbitrary constants, though all solutions of this form may be obtained with A non-negative and ϕ between $-\pi$ and π. This case is called *weak damping*.

(ii) $r^2 = 4mk$ (equal roots for λ)

The auxiliary equation has just one solution, $\lambda = -\rho$, where $\rho = r/(2m)$. So the differential equation has general solution

$$x(t) = Be^{-\rho t} + Cte^{-\rho t},$$

where B and C are arbitrary constants. This case is called *critical damping*.

(iii) $r^2 > 4mk$ (distinct real roots for λ)

The roots are $\lambda = -\rho_1$ and $\lambda = -\rho_2$, where

$$\rho_1 = \frac{r - \sqrt{r^2 - 4mk}}{2m} \quad \text{and} \quad \rho_2 = \frac{r + \sqrt{r^2 - 4mk}}{2m}.$$

Note that both ρ_1 and ρ_2 are positive. The general solution is

$$x(t) = Be^{-\rho_1 t} + Ce^{-\rho_2 t},$$

where B and C are arbitrary constants. This case is called *strong damping*.

9. (i) The most efficient way of finding the stiffness k is to use energy conservation (*Unit 7* Section 3), since there is no damping in operation between the firing of the gun and maximum spring compression. Equating the initial kinetic energy, with $m = 10^3$ and $v = 30$, to the potential energy for a compression of 2 metres, we obtain

$$\tfrac{1}{2} \times 10^3 \times 30^2 = \tfrac{1}{2}k \times 2^2$$

or $k = 2.25 \times 10^5\,\text{N}\,\text{m}^{-1}$.

(ii) For critical damping we require $r^2 = 4mk$, that is,

$$r = 2\sqrt{mk} = 2\sqrt{10^3 \times 2.25 \times 10^5} = 3 \times 10^4.$$

Thus a damping constant of $3 \times 10^4\,\text{N}\,\text{m}^{-1}\,\text{s}$ is required.

Solutions to the exercises in Section 2

1. We look for a particular (steady-state) solution of the form

$$x = B\cos\omega t + C\sin\omega t,$$

whose first two derivatives are

$$\dot{x} = -B\omega\sin\omega t + C\omega\cos\omega t,$$
$$\ddot{x} = -B\omega^2\cos\omega t - C\omega^2\sin\omega t.$$

Substitution of these expressions into the differential equation

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t$$

produces

$$m(-B\omega^2\cos\omega t - C\omega^2\sin\omega t) + r(C\omega\cos\omega t - B\omega\sin\omega t) + k(B\cos\omega t + C\sin\omega t) = P\cos\omega t.$$

We can now compare the coefficients of the cosine and sine terms in turn on both sides of this equation: the coefficient of $\cos\omega t$ on the left-hand side must equal P on the right, and the coefficient of $\sin\omega t$ on the left-hand side must equal zero (since there is no $\sin\omega t$ term on the right). This gives

$$-mB\omega^2 + rC\omega + kB = P, \tag{1}$$
$$-mC\omega^2 - rB\omega + kC = 0. \tag{2}$$

From Equation (2) we have $(k - m\omega^2)C = r\omega B$, or

$$C = \frac{r\omega B}{k - m\omega^2}. \tag{3}$$

Substituting for C in Equation (1), we obtain

$$(k - m\omega^2)B + \frac{r^2\omega^2}{k - m\omega^2}B = P$$

which leads to

$$B = \frac{P(k - m\omega^2)}{(k - m\omega^2)^2 + r^2\omega^2}.$$

It follows from Equation (3) that

$$C = \frac{Pr\omega}{(k - m\omega^2)^2 + r^2\omega^2}.$$

Having found the steady-state solution in the form

$$x = B\cos\omega t + C\sin\omega t,$$

it remains to translate this into the form

$$x = A\cos(\omega t + \phi),$$

where, from *Unit 7* Subsection 2.3,

$$A = \sqrt{B^2 + C^2},$$

and $\phi = -\arccos(B/A)$ (since $C > 0$).

Using the expressions above for B and C, we find

$$A = \frac{P}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}$$

and $$\phi = -\arccos\left(\frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}\right),$$

in agreement with Equations (3) and (4) of Section 1.

[Notice that this approach takes longer than the phasor method.]

2. (i) The angular frequency is

$$\omega = 2\pi f \quad (\text{where } f \text{ is frequency in Hz})$$
$$= 2\pi \times 2.5 \simeq 15.7\,\text{rad}\,\text{s}^{-1}.$$

(ii) From Equation (3) of Section 2, the amplitude is

$$A = \frac{P}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}.$$

With $m = 800$, $k = 5 \times 10^4$, $r = 9 \times 10^3$, $P = 800$ and $\omega \simeq 15.7$, we have

$$k - m\omega^2 \simeq -1.47 \times 10^5$$

and

$$A \simeq \frac{800}{\sqrt{(-1.47\times10^5)^2+(9\times10^3\times15.7)^2}}$$
$$\simeq \frac{800}{\sqrt{4.17\times10^{10}}} \simeq \frac{800}{2.04\times10^5} \simeq 3.92\times10^{-3}\,\mathrm{m},$$

so the amplitude of the vibration is about 4 mm.

(iii) From Equation (4) of Section 2, the phase angle of the vibration is

$$\phi = -\arccos\left(\frac{k-m\omega^2}{\sqrt{(k-m\omega^2)^2+r^2\omega^2}}\right).$$

Using the values above, we obtain

$$\phi \simeq -\arccos\left(\frac{-1.47\times10^5}{2.04\times10^5}\right)$$
$$\simeq -\arccos(-0.722)$$
$$\simeq -2.38\,\mathrm{rad}\quad(\text{or } -136^\circ).$$

The time lag is therefore

$$-\frac{\phi}{\omega} \simeq \frac{2.38}{15.7} \simeq 0.15\,\mathrm{s}.$$

3. The applied force is $\cos t = \mathrm{Re}(e^{it})$, with angular frequency $\omega = 1$. We look for a steady-state solution of the form $x = \mathrm{Re}(Ze^{it})$, where Z is a complex constant (namely, the phasor of x). Successive differentiations of this give

$$\dot{x} = \mathrm{Re}(iZe^{it}) \quad\text{and}\quad \ddot{x} = \mathrm{Re}(-Ze^{it}).$$

Substitution of these expressions into the equation of motion produces

$$\mathrm{Re}(-2Ze^{it}+5iZe^{it}+14Ze^{it}) = \mathrm{Re}(e^{it})$$

or $\quad \mathrm{Re}((12+5i)Ze^{it}) = \mathrm{Re}(e^{it}).$

This equation will be satisfied provided that

$$(12+5i)Z = 1,$$

for which we require

$$Z = \frac{1}{12+5i} = \frac{12-5i}{12^2+5^2} = \tfrac{12}{169} - \tfrac{5}{169}i = \tfrac{1}{13}e^{i\phi},$$

where

$$\phi = -\arccos(\tfrac{12}{13})$$
$$\simeq -0.395\,\mathrm{rad}\quad(\text{or } -22.6^\circ).$$

Hence the steady-state solution is

$$x = \mathrm{Re}\left(\tfrac{1}{13}e^{i\phi}e^{it}\right) = \mathrm{Re}\left(\tfrac{1}{13}e^{i(t+\phi)}\right)$$
$$= \tfrac{1}{13}\cos(t+\phi),$$

where $\phi \simeq -0.395$ rad.

4. **(i)** Using Euler's formula to expand $e^{i\omega t}$, we have

$$(M-iN)e^{i\omega t}$$
$$= (M-iN)(\cos\omega t + i\sin\omega t)$$
$$= (M\cos\omega t + N\sin\omega t) + i(M\sin\omega t - N\cos\omega t),$$

so that

$$\mathrm{Re}((M-iN)e^{i\omega t}) = M\cos\omega t + N\sin\omega t.$$

(ii) The applied force is (using the above)

$$2\cos 2t + 9\sin 2t = \mathrm{Re}((2-9i)e^{2it}).$$

The angular frequency of this applied force is $\omega = 2$, so we look for a steady-state solution of the form $x = \mathrm{Re}(Ze^{2it})$, whose first two derivatives are

$$\dot{x} = \mathrm{Re}(2iZe^{2it}) \quad\text{and}\quad \ddot{x} = \mathrm{Re}(-4Ze^{2it}).$$

Substituting these expressions into the equation of motion, we have

$$\mathrm{Re}(-4Ze^{2it}+4iZe^{2it}+5Ze^{2it}) = \mathrm{Re}((2-9i)e^{2it})$$

or $\quad \mathrm{Re}((1+4i)Ze^{2it}) = \mathrm{Re}((2-9i)e^{2it}).$

This equation will be satisfied if

$$(1+4i)Z = (2-9i),$$

for which we require

$$Z = \frac{2-9i}{1+4i} = \frac{(2-9i)(1-4i)}{1^2+4^2} = -2-i.$$

Using the result of part (i) once more, the steady-state solution is

$$x = \mathrm{Re}((-2-i)e^{2it})$$
$$= -2\cos 2t + \sin 2t.$$

To find the amplitude A and phase ϕ of this vibration, we express the phasor Z in its exponential form $Z = Ae^{i\phi}$. This gives

$$A = \sqrt{(-2)^2+1^2} = \sqrt{5} \simeq 2.24\,\mathrm{m}$$

and $\quad \phi = -\arccos(-2/\sqrt{5}) \simeq -2.68\,\mathrm{rad},$

so the amplitude and phase are respectively 2.24 m and −2.68 rad.

5. We seek the phasor Z for the steady-state solution of

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t = \mathrm{Re}\left(\frac{P}{m}e^{i\omega t}\right).$$

Putting $x = \mathrm{Re}(Ze^{i\omega t})$ leads to $\dot{x} = \mathrm{Re}(i\omega Ze^{i\omega t})$ and $\ddot{x} = \mathrm{Re}(-\omega^2 Ze^{i\omega t})$, so the phasor Z is given by

$$(-\omega^2 + i2\alpha\omega_0\omega + \omega_0^2)Z = P/m$$

or $\quad Z = \dfrac{P/m}{\omega_0^2-\omega^2+i2\alpha\omega_0\omega}$

$$= \frac{P(\omega_0^2-\omega^2-i2\alpha\omega_0\omega)}{m((\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2)}.$$

Hence the amplitude of the steady-state solution is

$$A = |Z| = \frac{P\sqrt{(\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2}}{m((\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2)}$$
$$= \frac{P/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2}},$$

and the phase is

$$\phi = \mathrm{Arg}(Z) = -\arccos\left(\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2}}\right).$$

6. Dividing each coefficient in the differential equation by the coefficient of $\ddot{x}$ (so that the coefficient of $\ddot{x}$ becomes unity), we obtain

$$\ddot{x} + \tfrac{9}{4}\dot{x} + 25x = \tfrac{9}{4}\cos 6t.$$

We now compare the coefficients of this equation with those of Equation (10) in Section 2, from which it follows that

$$2\alpha\omega_0 = \tfrac{9}{4},\quad \omega_0^2 = 25,\quad P/m = \tfrac{9}{4}.$$

We also deduce from the two right-hand sides that $\omega = 6$.

(i) From above, the undamped angular frequency is $\omega_0 = 5$, so that the damping ratio is

$$\alpha = \frac{9}{8\omega_0} = \frac{9}{40} = 0.225.$$

(ii) The amplitude of the steady-state solution is

$$A = \frac{P/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2}}$$
$$= \frac{2.25}{\sqrt{(25-36)^2+(2.25\times6)^2}}$$
$$= \frac{2.25}{\sqrt{121+182.25}} \simeq 0.129\,\mathrm{m},$$

and the phase is

$$\phi = -\arccos\left(\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\alpha^2\omega_0^2\omega^2}}\right)$$
$$= -\arccos\left(\frac{-11}{\sqrt{121+182.25}}\right)$$
$$\simeq -2.255\,\mathrm{rad}\quad(\text{or } -129.2^\circ).$$

The angular frequency of the steady-state solution is $\omega = 6\,\mathrm{rad\,s^{-1}}$.

7. From Solution 6 above, we have

$$\frac{P}{m} = \frac{9}{4}, \quad \frac{k}{m} = \omega_0^2 = 25, \quad A \simeq 0.129.$$

Hence $k/P = 100/9$ and

$$M = \frac{Ak}{P} \simeq \frac{0.129 \times 100}{9} \simeq 1.44.$$

8. The relative error will be less than 10% provided that M is between 0.9 and 1.1, which from the graph for $\alpha = 0.6$ appears to be the case for the approximate range $0 \leq \omega/\omega_0 \leq 0.9$. Since $0.9\omega_0 = 0.9 \times 20 = 18$, the corresponding frequency range (for ω) is between 0 and $18\,\mathrm{rad\,s^{-1}}$.

9. We use the values $m = 800$, $k = 5 \times 10^4$, $r = 9 \times 10^3$ and $P = 800$ given in Exercise 2, together with the value $\omega \simeq 15.7$ obtained in Solution 2(i). The undamped angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 10^4}{800}} \simeq 7.91\,\mathrm{rad\,s^{-1}},$$

and the damping ratio is

$$\alpha = \frac{r}{2\sqrt{mk}} = \frac{9 \times 10^3}{2\sqrt{800 \times 5 \times 10^4}} \simeq 0.712.$$

[Note that this value of α is just above $1/\sqrt{2}$, so that there is no input frequency which makes the car body resonate.]

The relevant value of ω/ω_0 is approximately $\frac{15.7}{7.91} \simeq 2.0$. The corresponding value of M from Figure 5(a) in Section 2 is approximately 0.25. Since $M = Ak/P$, we then have

$$A = \frac{MP}{k} \simeq \frac{0.25 \times 800}{5 \times 10^4} = 4 \times 10^{-3}.$$

The estimated value of ϕ from Figure 5(b) is

$$\phi \simeq -0.75\pi \simeq -2.36\,\mathrm{rad}.$$

The estimated values of A and ϕ here are both close to the values calculated in Exercise 2.

10. With $p = \omega/\omega_0$, Equation (13) of Section 2 becomes

$$M = \frac{1}{\sqrt{(1-p^2)^2 + 4\alpha^2 p^2}}.$$

Differentiation with respect to p gives

$$\frac{dM}{dp} = \frac{-\frac{1}{2}[2(1-p^2)(-2p) + 8\alpha^2 p]}{[(1-p^2)^2 + 4\alpha^2 p^2]^{3/2}},$$

so that $dM/dp = 0$ (for points on the graph with horizontal slope) when

$$-4p(1-p^2) + 8\alpha^2 p = 0$$

or $\quad p(p^2 - 1 + 2\alpha^2) = 0.$

The roots of this cubic equation are

$$p = 0 \quad \text{and} \quad p = \pm\sqrt{1 - 2\alpha^2}.$$

(i) We know that $p = \omega/\omega_0$ is non-negative. For $\alpha < 1/\sqrt{2}$, the non-negative roots are $p = 0$ and $p = \sqrt{1-2\alpha^2}$, and the graphs in Figure 5(a) of Section 2 indicate that the second of these gives a maximum value for M. (This can be confirmed by examining the sign of dM/dp to the left and to the right of $p = \sqrt{1-2\alpha^2}$.)

(ii) For $\alpha = 1/\sqrt{2}$, the root $p = 0$ is repeated, and for $\alpha > 1/\sqrt{2}$, the expression $\sqrt{1-2\alpha^2}$ gives complex roots. Figure 5(a) shows that in this case M is a decreasing function of p for $p > 0$, so that $p = 0$ gives the maximum value for M. (It can also be shown from above that $dM/dp < 0$ for $p > 0$ and $\alpha \geq 1/\sqrt{2}$.)

Solutions to the exercises in Section 3

1.

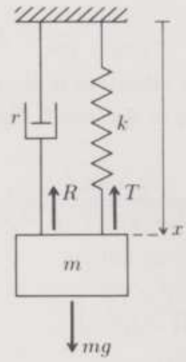


Figure 1

In deriving the equation of motion we assume that the spring is extended and that the dashpot's length is increasing. The corresponding force directions are as shown above.

The extension of the spring is $x - l_0$, and the rate of increase of the dashpot's length is $\dot{x}$. The spring and dashpot forces therefore have respective magnitudes

$$T = k(x - l_0) \quad \text{and} \quad R = r\dot{x}.$$

The only other force acting on the particle is the gravitational force, of magnitude mg vertically downwards. Newton's second law therefore gives

$$\begin{aligned} m\ddot{x} &= mg - T - R \\ &= mg - k(x - l_0) - r\dot{x} \end{aligned}$$

or $\quad m\ddot{x} + r\dot{x} + kx = mg + kl_0.$

This equation of motion is identical to Equation (3) in Section 1, as expected (since the model considered is the same in the two cases).

2.

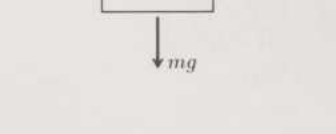


Figure 2

We assume, as usual, that the spring is in tension and that the dashpot is lengthening. The corresponding force diagram is shown above. Note that $P\cos\omega t$ is the x-component of the external force, so that this force has the direction shown only when $P\cos\omega t$ is positive (the same applies to the diagram given in the question).

The spring force has magnitude $T = k(x - l_0)$ and the dashpot force has magnitude $R = r\dot{x}$, so that the required equation of motion is

$$m\ddot{x} = P\cos\omega t - r\dot{x} - k(x - l_0)$$

or $\quad m\ddot{x} + r\dot{x} + kx = kl_0 + P\cos\omega t.$

3. We seek the phasor $Z = Ae^{i\phi}$, where $x = \mathrm{Re}(Ze^{i\omega t})$ is the steady-state solution of the differential equation

$$m\ddot{x} + r\dot{x} + kx = A_0(k\cos\omega t - r\omega\sin\omega t).$$

Substituting for x and its derivatives, this equation can be written as

$$\mathrm{Re}[(-m\omega^2 + ir\omega + k)Ze^{i\omega t}] = \mathrm{Re}[A_0(k + ir\omega)e^{i\omega t}],$$

where the result of Exercise 4(i) of Section 2 has been used on the right-hand side. This equation will be satisfied if

$$(-m\omega^2 + ir\omega + k)Z = A_0(k + ir\omega)$$

or

$$Z = \frac{A_0(k + ir\omega)}{(k - m\omega^2) + ir\omega}$$
$$= \frac{A_0(k + ir\omega)[(k - m\omega^2) - ir\omega]}{(k - m\omega^2)^2 + r^2\omega^2}$$
$$= \frac{A_0\left([k(k - m\omega^2) + r^2\omega^2] - imr\omega^3\right)}{(k - m\omega^2)^2 + r^2\omega^2}.$$

This is of the form $Z = u - iv$, where

$$u = \frac{A_0[k(k - m\omega^2) + r^2\omega^2]}{(k - m\omega^2)^2 + r^2\omega^2}$$

and

$$v = \frac{A_0 mr\omega^3}{(k - m\omega^2)^2 + r^2\omega^2},$$

so that the amplitude is

$$A = |Z| = \sqrt{u^2 + v^2}$$
$$= \frac{A_0\sqrt{[k(k - m\omega^2) + r^2\omega^2]^2 + (mr\omega^3)^2}}{(k - m\omega^2)^2 + r^2\omega^2}$$

and the phase is

$$\phi = -\arccos(u/|Z|)$$
$$= -\arccos\left(\frac{k(k - m\omega^2) + r^2\omega^2}{\sqrt{[k(k - m\omega^2) + r^2\omega^2]^2 + (mr\omega^3)^2}}\right).$$

4. **(i)** When the particle is in equilibrium, the dashpot length is constant and so the dashpot exerts no force. The force diagram is therefore as shown below, where T is the magnitude of the spring force.

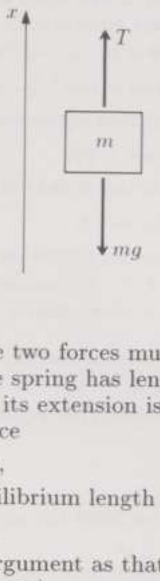


Figure 3

In equilibrium these two forces must balance, so that $T = mg$. Now if the spring has length l_1 when the particle is in equilibrium then its extension is $l_1 - l_0$, giving $T = k(l_1 - l_0)$. Hence

$$k(l_1 - l_0) = mg,$$

from which the equilibrium length of the spring is

$$l_1 = l_0 + mg/k.$$

(This is the same argument as that required to answer Exercise 1 of Section 1.)

(ii)

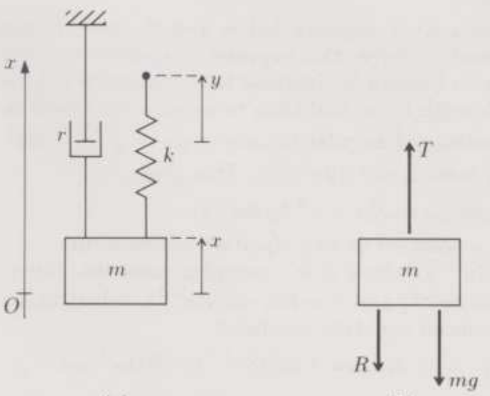


Figure 4 (a) (b)

Figure 4(a) shows the system when displaced from its rest position. The displacements x of the particle and y of the forcing point are as described in the question. Figure 4(b) shows the forces which act on the particle when the spring is extended (as it must always be if a rope is being modelled) and the particle is moving upwards (that is, the dashpot is contracting). The magnitudes of the spring and dashpot forces are denoted by T and R respectively.

The extension of the spring is

$$l_0 + mg/k + y - x - l_0,$$

so that

$$T = k(mg/k + y - x).$$

Since $\dot{x} > 0$, we also have $R = r\dot{x}$. Application of Newton's second law gives

$$m\ddot{x} = T - R - mg$$
$$= k(mg/k + y - x) - r\dot{x} - mg,$$

or $\quad m\ddot{x} + r\dot{x} + kx = ky.$

This is the required equation of motion.

(iii) Putting $y = A_0 \cos \omega t$ in the result of part (ii) gives the equation of motion

$$m\ddot{x} + r\dot{x} + kx = kA_0 \cos \omega t.$$

This is the same as Equation (1) of Section 2, except for the presence of kA_0 rather than P on the right-hand side. The graphs in Figure 5 of Section 2 can therefore be applied to this situation where, with kA_0 in place of P, we have $M = A/A_0$. Note that M here is the ratio of the displacement amplitudes of the particle (output) and of the upper end of the spring (input). (This quantity is called the *amplitude ratio* of the system.)

From the values given in the question, we obtain

$$\omega_0 = \sqrt{k/m} = \sqrt{50} \simeq 7.07\,\mathrm{rad\,s^{-1}},$$
$$\omega = 2 \times 2\pi \simeq 12.6\,\mathrm{rad\,s^{-1}},$$

and hence $\omega/\omega_0 \simeq 1.78$. Corresponding to this value of ω/ω_0, the graph for $\alpha = 0.2$ in Figure 5(a) of Section 2 gives $M \simeq 0.45$, from which

$$A \simeq 0.45A_0 = 0.45 \times 0.05 \simeq 0.023.$$

The amplitude of the particle's steady-state oscillations is therefore about 0.023 m, or 2.3 cm.

[We may check that the spring representing the rope is in tension throughout the motion by showing that T (as defined above) is always positive. Since $|x| \leq A$ and $|y| \leq A_0$, we have

$$T = mg + k(y - x) \geq mg - k(A_0 + A).$$

The value of the right-hand expression is approximately

$$4 \times 9.8 - 200(0.05 + 0.023) \simeq 24.6\,\mathrm{N},$$

so the spring is always in tension.]

Solutions to the exercises in Section 4

1.

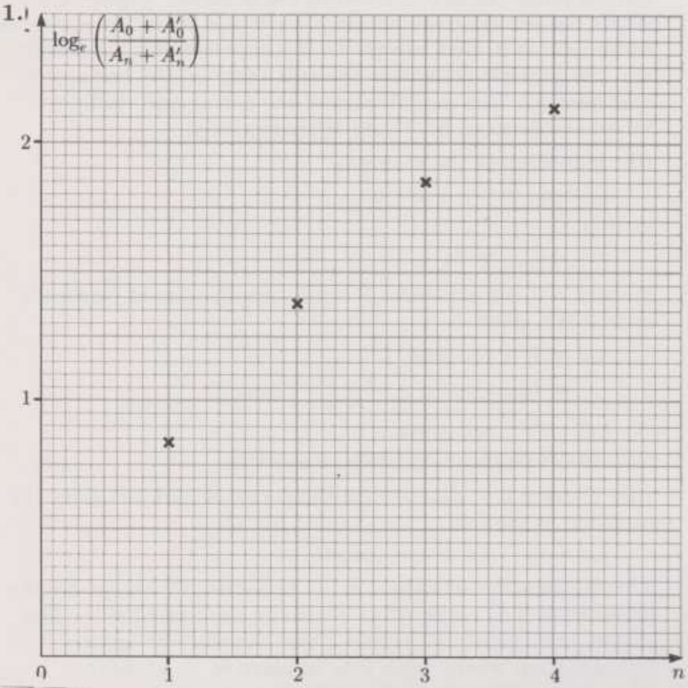


Figure 1

The model predicts that a graph of $\log_e[(A_0+A_0')/(A_n+A_n')]$ against n should be a straight line through the origin (with slope $\rho\tau$). The given data have been plotted on such a graph above. There is no straight line through the origin which provides a convincing fit to these data points, so this test indicates that linear damping is not a satisfactory modelling assumption for a water-filled dashpot.

2. **(i)** Assume that the spring is extended and that the dashpot is lengthening. The spring force is then of magnitude $T = k(x-y-l_0)$ and the dashpot force is of magnitude $R = r\dot{x}$, both being directed downwards. The only other force acting is that due to gravity, which is of magnitude mg directed downwards. Newton's second law therefore gives

$$\begin{aligned} m\ddot{x} &= -T - R - mg \\ &= -k(x-y-l_0) - r\dot{x} - mg, \end{aligned}$$

or $\quad m\ddot{x} + r\dot{x} + kx = kl_0 - mg + ky,$

which is Equation (1) of the text.

(ii) Equation (2) of the text is

$$m\ddot{x} + r\dot{x} + kx = k(l_0 + y_0 - mg/k) + kA_0\cos\omega t.$$

The steady-state solution is

$$x = l_0 + y_0 - mg/k + A\cos(\omega t + \phi),$$

where the sinusoidal term is the steady-state solution of the equation

$$m\ddot{x} + r\dot{x} + kx = kA_0\cos\omega t.$$

We now proceed much as in Solution 4(iii) of Section 3. The last equation is the same as Equation (1) of Section 2, with P replaced by kA_0. The analysis of Subsection 2.3 can therefore be applied, leading to Equation (13) of Section 2, which was

$$M = \frac{1}{\sqrt{(1-(\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}}.$$

Here M is equal to A/A_0, since P has been replaced by kA_0. This is the expression for the amplitude ratio given in Equation (3) of Section 4.

(iii) From the result of Exercise 10 of Section 2, there will be no resonance provided that the damping ratio α is greater than or equal to $1/\sqrt{2}$.

3. **(i)**

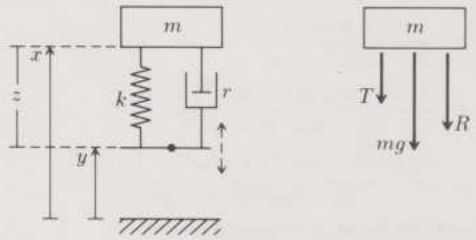


Figure 2

We assume that the spring is extended and that the dashpot is lengthening. Then the spring and dashpot forces are as indicated in the diagram. The lengths of the spring and dashpot are both $x-y$. Hence the extension of the spring is $x-y-l_0$, and the rate of increase of the dashpot's length is $\dot{x}-\dot{y}$. The magnitudes of the spring and dashpot forces are therefore

$$T = k(x-y-l_0) \quad \text{and} \quad R = r(\dot{x}-\dot{y})$$

respectively. So the equation of motion of the particle is

$$\begin{aligned} m\ddot{x} &= -T - R - mg \\ &= -k(x-y-l_0) - r(\dot{x}-\dot{y}) - mg \end{aligned}$$

or $\quad m\ddot{x} + r\dot{x} + kx = kl_0 - mg + r\dot{y} + ky.$

(ii) Since $z = x - y$, we have

$$\dot{z} = \dot{x} - \dot{y} \quad \text{and} \quad \ddot{z} = \ddot{x} - \ddot{y}.$$

The equation of motion becomes

$$m(\ddot{z}+\ddot{y}) + r(\dot{z}+\dot{y}) + k(z+y) = kl_0 - mg + r\dot{y} + ky$$

or $\quad m\ddot{z} + r\dot{z} + kz = kl_0 - mg - m\ddot{y}.$

4. The equation of motion of the tone-arm in terms of $z = x - y$ is given by the last equation in Solution 3 above. With $y = A_0\cos\omega t$ we have $\ddot{y} = -\omega^2 A_0\cos\omega t$, giving

$$m\ddot{z} + r\dot{z} + kz = kl_0 - mg + m\omega^2 A_0\cos\omega t.$$

The steady-state solution of this equation is

$$z = l_0 - mg/k + A\cos(\omega t + \phi),$$

where the sinusoidal term is the steady-state solution of

$$m\ddot{z} + r\dot{z} + kz = m\omega^2 A_0\cos\omega t. \tag{1}$$

This is, once again, the same as Equation (1) of Section 2, but with $m\omega^2 A_0$ in place of P. As before we may therefore conclude that Equation (13) of Section 2 holds, which is

$$M = \frac{1}{\sqrt{(1-(\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}}.$$

Here $M = Ak/P$ (from Section 2) is replaced by

$$M = \frac{Ak}{m\omega^2 A_0} = \frac{\omega_0^2 A}{\omega^2 A_0},$$

since $\omega_0 = \sqrt{k/m}$. Hence we have the amplitude ratio

$$\frac{A}{A_0} = \frac{(\omega/\omega_0)^2}{\sqrt{(1-(\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}},$$

as required.

As an alternative, we show below how the phasor method can be used to derive this expression directly, without reference to Section 2. Starting from Equation (1) above, we divide through by m and then re-express the result in terms of the undamped angular frequency $\omega_0 = \sqrt{k/m}$ and the damping ratio $\alpha = r/(2\sqrt{mk})$. This gives

$$\ddot{z} + 2\alpha\omega_0\dot{z} + \omega_0^2 z = \omega^2 A_0\cos\omega t.$$

We seek a solution to this equation of the form $z = \mathrm{Re}(Ze^{i\omega t})$, where Z is a complex constant. Since $\dot{z} = \mathrm{Re}(i\omega Ze^{i\omega t})$ and $\ddot{z} = \mathrm{Re}(-\omega^2 Ze^{i\omega t})$, substitution into the differential equation produces

$$\mathrm{Re}[(-\omega^2 + 2i\alpha\omega_0\omega + \omega_0^2)Ze^{i\omega t}] = \mathrm{Re}(\omega^2 A_0 e^{i\omega t}).$$

This is satisfied provided that

$$(-\omega^2 + 2i\alpha\omega_0\omega + \omega_0^2)Z = \omega^2 A_0,$$

for which we require

$$Z = \frac{\omega^2 A_0}{\omega_0^2 - \omega^2 + 2i\alpha\omega_0\omega} = \frac{(\omega/\omega_0)^2 A_0}{1 - (\omega/\omega_0)^2 + 2i\alpha\omega/\omega_0}.$$

The amplitude of these steady-state vibrations is

$$A = |Z| = \frac{(\omega/\omega_0)^2 A_0}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}},$$

giving the required amplitude ratio as before.

Solutions to the exercises in Section 5

1. **(i)** We model the object by a particle of mass m, and assume that the spring and dashpots are perfect, with the spring having stiffness k and each dashpot having dashpot constant r. The particle displacement x is measured downwards from the equilibrium position. This model system is shown on the left below.

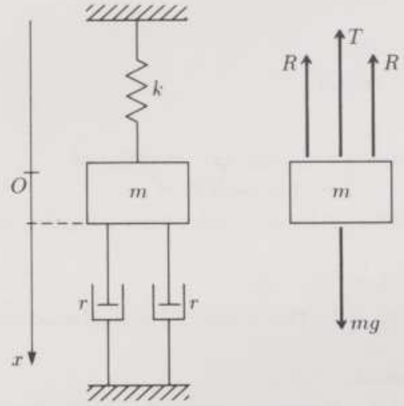


Figure 1

Assuming that the spring is extended and the dashpots are contracting, the forces acting on the particle are as shown on the right of the diagram above. The magnitudes of the spring and dashpot forces are respectively

$$T = mg + kx$$

(since $T = mg$ when the particle is in equilibrium) and

$$R = r|\dot{x}| = r\dot{x}.$$

Newton's second law then gives

$$m\ddot{x} = mg - T - R - R = mg - (mg + kx) - 2r\dot{x}$$

or $m\ddot{x} + 2r\dot{x} + kx = 0.$

This can be written as

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = 0,$$

where $\omega_0 = \sqrt{k/m}$ is the undamped angular frequency and $\alpha = r/\sqrt{mk}$ is the total effective damping ratio (that is, the ratio obtained by regarding all the damping as being produced by a single dashpot with dashpot constant $2r$). In the equilibrium position, a force of magnitude mg produces a spring extension of 19.6 cm = 0.196 m, so that

$$mg = 0.196k$$

or $\omega_0^2 = \dfrac{k}{m} = \dfrac{g}{0.196} = 50.$

Also $\alpha = 1$ is given, so the equation of motion is

$$\ddot{x} + 2\sqrt{50}\dot{x} + 50x = 0.$$

(ii) Removing one dashpot has the effect of halving the damping. The new equation of motion is therefore

$$\ddot{x} + \sqrt{50}\dot{x} + 50x = 0,$$

where the damping ratio is now $\alpha = r/(2\sqrt{mk}) = 0.5$. From Equations (7) and (8) of Section 1, the general solution of this equation is

$$x = Ae^{-\rho t}\cos(\Omega t + \phi), \tag{1}$$

where

$$\rho = \frac{r}{2m}, \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m},$$

and A, ϕ are arbitrary constants. From $\omega_0^2 = 50$ and $\alpha = 0.5$, we have

$$k = 50m \quad \text{and} \quad r = \sqrt{mk} = \sqrt{50}m,$$

giving

$$\rho = \sqrt{12.5} \quad \text{and} \quad \Omega = \tfrac{1}{2}\sqrt{200 - 50} = \sqrt{37.5}. \tag{2}$$

The period of damped vibrations is

$$\tau = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{37.5}} \simeq 1.03\,\text{s}.$$

If the particle starts from rest with the spring at its natural length then, since the origin is at the particle's equilibrium position, the initial conditions are

$$x(0) = -0.196 \quad \text{and} \quad \dot{x}(0) = 0.$$

Putting $t = 0$ into Equation (1) and applying the first of these conditions produces

$$A\cos\phi = -0.196.$$

The maximum subsequent spring extension occurs after half a period at time $t = \frac{1}{2}\tau = \pi/\Omega$, when $\dot{x}(t) = 0$ once more. The corresponding value of x is (using Equations (2))

$$x(\pi/\Omega) = Ae^{-\rho\pi/\Omega}\cos(\pi + \phi) = -Ae^{-\rho\pi/\Omega}\cos\phi = 0.196e^{-\pi/\sqrt{3}} \simeq 0.032.$$

The spring extension is then approximately

$$0.196 + 0.032 = 0.228\,\text{m},$$

or approximately 22.8 cm.

2. The equation of motion for Exercise 1(ii) was

$$m\ddot{x} + r\dot{x} + kx = 0,$$

and with the additional downward magnetic force, this becomes

$$m\ddot{x} + r\dot{x} + kx = P_0 + P\cos\omega t.$$

The steady-state solution is

$$x = P_0/k + A\cos(\omega t + \phi),$$

where the sinusoidal term is the steady-state solution of the equation

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t,$$

that is,

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t, \tag{3}$$

where $\omega_0 = \sqrt{k/m}$ and $\alpha = r/(2\sqrt{mk})$. Equation (3) is identical to Equation (10) of Section 2, and the corresponding amplitude of forced vibrations is given by Equation (11) of Section 2 as

$$A = \frac{P/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}.$$

Now $\alpha = 0.5$ from Exercise 1(ii), so the amplitude is

$$A = \frac{P/(m\omega_0^2)}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + (\omega/\omega_0)^2}} = \frac{d}{\sqrt{(1 - p^2)^2 + p^2}},$$

where $p = \omega/\omega_0$ and $d = P/(m\omega_0^2) = P/k$ is the distance by which a downward force of constant magnitude P extends the spring.

3. **(i)** Let x be the upward displacement of the particle, measured from the fixed point about which the sinusoidal motion $y = A_0 \cos \omega t$ of the base of the box takes place. Then we have $x = y + z + d$, where d is the length of the dashpot in the equilibrium position. Assuming that the spring is extended and the dashpot lengthening, the forces acting on the particle are as shown in the diagram below.

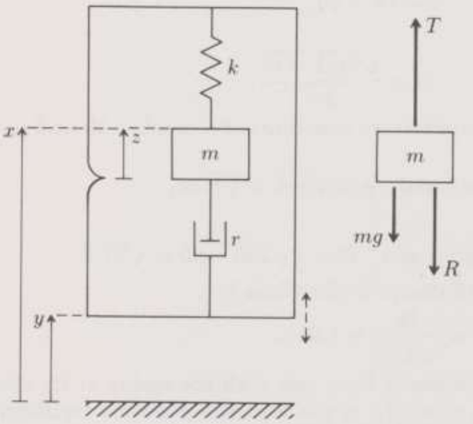


Figure 2

The spring and dashpot forces are respectively

$$T = mg - kz$$

(since $T = mg$ when the particle is in equilibrium) and

$$R = r\dot{z}.$$

Newton's second law therefore gives

$$\begin{aligned} m\ddot{x} &= T - mg - R \\ &= (mg - kz) - mg - r\dot{z}. \end{aligned}$$

On putting $x = y + z + d$ we have $\ddot{x} = \ddot{y} + \ddot{z}$, so the equation of motion relative to the box is

$$m\ddot{z} + r\dot{z} + kz = -m\ddot{y}$$

or $\quad m\ddot{z} + r\dot{z} + kz = m\omega^2 A_0 \cos \omega t.$

(ii) The last equation is a copy of Equation (1) of Section 2, with $m\omega^2 A_0$ in place of P, so by Equations (11) and (12) of that section we have the steady-state solution

$$z = A\cos(\omega t + \phi),$$

where

$$A = \frac{\omega^2 A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}$$

and $\quad \phi = -\arccos\left(\dfrac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}\right).$

(iii) If ω_0/ω and α are both small, then

$$A = \frac{A_0}{\sqrt{((\omega_0/\omega)^2 - 1)^2 + 4\alpha^2(\omega_0/\omega)^2}} \simeq A_0$$

and

$$\begin{aligned} \phi &= -\arccos\left(\frac{(\omega_0/\omega)^2 - 1}{\sqrt{((\omega_0/\omega)^2 - 1)^2 + 4\alpha^2(\omega_0/\omega)^2}}\right) \\ &\simeq -\arccos(-1) = -\pi. \end{aligned}$$

Hence we have

$$\begin{aligned} z &= A\cos(\omega t + \phi) \\ &\simeq A_0\cos(\omega t - \pi) = -A_0\cos\omega t \end{aligned}$$

or $\quad z \simeq -y.$

4. **(i)** If ω/ω_0 and α are both small, then

$$A = \frac{(\omega/\omega_0)^2 A_0}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}} \simeq \frac{\omega^2 A_0}{\omega_0^2}$$

and

$$\begin{aligned} \phi &= -\arccos\left(\frac{1 - (\omega/\omega_0)^2}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}}\right) \\ &\simeq -\arccos(1) = 0. \end{aligned}$$

Hence we have

$$\begin{aligned} z &= A\cos(\omega t + \phi) \\ &\simeq (\omega/\omega_0)^2 A_0 \cos\omega t \end{aligned}$$

or $\quad z \simeq -\ddot{y}/\omega_0^2.$

Thus z is proportional to $\ddot{y}$, and the constant of proportionality $-1/\omega_0^2$ is independent of ω.

(ii) The approximate value of acceleration as indicated by the instrument is

$$-\omega_0^2 z = -\omega_0^2 A\cos(\omega t + \phi),$$

whose amplitude is $\omega_0^2 A$. The actual value of the acceleration is

$$\ddot{y} = -\omega^2 A_0 \cos\omega t,$$

with amplitude $\omega^2 A_0$. Using the expression above for A and putting $\alpha = \omega/\omega_0 = 0.1$, the ratio of these amplitudes is

$$\begin{aligned} \frac{\omega_0^2 A}{\omega^2 A_0} &= \frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + 4\alpha^2(\omega/\omega_0)^2}} \\ &= \frac{1}{\sqrt{(1 - (0.1)^2)^2 + 4 \times 0.1^4}} \\ &\simeq 1.01. \end{aligned}$$

The Open University
0 7492 2032 5